Polynomial Rings

Definition:- since we are familier with the polynomials. Therefore we know that the expression of the type

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n, \; a_n \neq 0$$

is called a polynomial of degree n and the variable x is called indeterminate.

Definition:- let R be a ring and x be an indeterminate which does not belong to R.

Then a polynomial of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n, \; a_n \neq 0$$

where $a_0, a_1, a_2, \dots, a_n \dots$ all are in R with finite and non-zero is called a polynomial over a ring R.

SET OF ALL POLYNOMIALS OVER A RING

let R be a ring and x be an indeterminate, then the set of all polynomials of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

where $a_0, a_1, a_2, \dots, a_n \dots$ all are in R with finite number of non-zero elements is called the set of polynomials over R and it is denoted by $R[x]$.

Zero polynomial :- A polynomial over a ring R of the type

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

is called a zero polynomial if $a_n = 0 \; \forall n$.

Equality, Sum and Product of Polynomial :-

Equality . Let R be a ring and let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

and $g(x) = b_0 + b_1 x + b_2 x^2 + \dots$

be two polynomials over R. Then they are said to be equal if $a_n = b_n \; \forall n$.

Sum of Polynomials :- Let R be a ring and x be indeterminate and let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

and $g(x) = b_0 + b_1 x + b_2 x^2 + \dots$

be two polynomials over R. Then they are said to be equal if $a_n = b_n \; \forall n$

$f(x) + g(x) = c_0 + c_1 x + c_2 x^2 + \dots$ is said to be the sum of $f(x)$ and $g(x)$ if $c_n = a_n + b_n \; \forall n$.

Product of Polynomials :- Let R be a ring and x be an indeterminate and let $f(x), g(x) \in R[x]$, where

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

and

$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

Then the polynomial

$$f(x) g(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

is said to be the product of $f(x)$ and $g(x)$ if

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$$

$$= \sum a_i b_j \text{ for all } n \in \text{(set of all non-negative integers)}$$

Degree of Polynomial Let $f(x) \in R[x]$ is of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

Then the degree of $f(x)$ is n if and only if $a_n \neq 0$ and $a_m = 0 \ \forall \ m > n$.

Definition :- If a polynomial $f(x)$ is of degree n, then the term $a_n x^n$ is called the leading term and a_n is called the leading Coefficient and a_0 is called the Constant term.