Abstract Algebra

Polynomial Ring

Definition The expressions of the type $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, $a_n \neq 0$ is called a polynomial of degree n and the variable x is called indeterminate.

Zero polynomial A polynomial over a ring R of the type

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

is called a zero polynomial if $a_n = 0$, $\forall n$.

Ideals

Definition An additive subgroup S of a ring R is said to be a left ideal of R iff $a \in S$, $r \in R \Rightarrow ra \in S$.

Definition A non empty subset S of a ring R is said to be an ideal of R if it is both left and right ideal. That is an additive subgroup S of R is an ideal of R if $a \in S$, $r \in R \Rightarrow ra \in S$, $ar \in S$.

Note! - (i) The necessary and sufficient conditions for a non-empty subset S of a rings R to be an ideal of R are

(a) $a \in S, b \in S \Rightarrow a-b \in S$.

(b) $a \in S, \quad r \in R \Rightarrow ra \in S$, and $ar \in S$

(ii) The intersection of any two ideals of a ring R is again an ideal of R.

(iii) If S is any ideal of a ring R and T any subring of R, then S is an ideal of $S+T$.

(iv) The intersection of all ideals of a ring R containing a non-empty subset M of R is the smallest ideal of R containing M.

(v) A field has no proper ideal.

(vi) If R is a commutative ring and $a \in R$, then the set $Ra = \{ra : r \in R\}$ is an ideal of R.

(vii) If R be a commutative ring with unit element whose only ideals are $\{0\}$ and R itself. Then R is a field.

Homomorphism of Rings

Definition Let R and R' be two rings with two binary operations addition and multiplication, then a mapping ϕ from the ring R into the ring R' is said to be a homomorphism if

(i) $\phi(a+b) = \phi(a) + \phi(b)$

(ii) $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R$

Notes :-

(1) Here both rings have same binary operations, in case both have the operation as $[R, '+', '\cdot']$ and $[R, '*', 'o']$ then above condition (i) and (ii) can be written as

(i) $\phi(a+b) = \phi(a) * \phi(b)$

(ii) $\phi(ab) = \phi(a) \circ \phi(b)$

(2) If the ring R is identical with R', then ϕ is known as ring endomorphism.

(3) If R and R' are identical and ϕ is bijective, then ϕ is known as ring automorphism.