Picard's Method of Successive Approximations:-

Integrating the given differential Equation

$$\frac{dy}{dx} = f(x,y) \text{ with the initial condition } y(x_0) = y_0 \quad \text{--- (1)}$$

we obtain

$$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx \quad \text{--- (2)}$$

In this Equation (2) the unknown function y appears under the integral sign is called integral equation. Such an equation can be solved by the method of successive approximations in which the first approximation to y is obtained by putting y_0 for y on right side of Eq (2) and we write

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x,y_0)\,dx$$

The integral on the right can now be solved and the resulting $y^{(1)}$ is substituted for y in the integrand of equation (2) to obtain the second approximation $y^{(2)}$. Thus

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\,dx$$

proceeding in this way, we obtain $y^{(3)}, y^{(4)} \dots y^{(n)}$ where.

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\,dx \quad \text{with } y^{(0)} = y_0 \quad \text{--- (3)}$$

Hence this method yields a sequence of approximations $y^{(1)}, y^{(2)}, \dots y^{(n)}$ and it can be proved that if the function $f(x,y)$ is bounded in some region about the point (x_0, y_0) and if $f(x,y)$ satisfies the Lipschitz condition, viz.

$$|f(x,y) - f(x,\bar{y})| \le K(|y-\bar{y}|), \quad K \text{ is constant}$$

then the sequence $y^{(1)}, y^{(2)}, y^{(3)} \dots$ converges to the solution of equation (1).

Exp. Solve the equation $\frac{dy}{dx} = y' = x + y^2$, with the condition $y = 1$ when $x = 0$

Solution We start with $y^{(0)} = 1$ and obtain

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\, dx$$

$$y^{(1)} = 1 + \int_0^x [x + 1]\, dx$$

$$y^{(1)} = 1 + x + \frac{1}{2}x^2$$

Then the second approximation is

$$y^{(2)} = y_0 + \int_0^x \left[x + \left(1 + x + \frac{x^2}{2}\right)^2\right] dx$$

$$= 1 + \int_0^x \left[x + 1 + x^2 + \frac{x^4}{4} + 2x + 2\cdot\frac{x^2}{2} + 2\cdot\frac{x^3}{2}\right] dx$$

$$= 1 + \int_0^x \left[1 + 3x + 2x^2 + x^3 + \frac{x^4}{4}\right] dx$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{x^4}{4} + \frac{1}{4}\cdot\frac{x^5}{5}$$

$$y^{(2)} = 1 + x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5$$

It is obvious that the integrations might become more and more difficult as we proceed to higher approximations.

Exp. Given the differential equation

$$\frac{dy}{dx} = \frac{x^3}{y^2+1}$$

with the initial condition $y=0$ when $x=0$ use Picard's method to obtain y for $x=0.25, 0.5$ and 1.0 correct to three decimal places.

Solution Given that

$$\frac{dy}{dx} = \frac{x^2}{y^2+1}$$

We have by integrating.

$$y = y_0 + \int_0^x \frac{x^2}{y^2+1}\,dx$$

$$y = 0 + \int_0^x \frac{x^2}{y^2+1}\,dx$$

Setting $y^{(0)} = 0$ then we obtain

$$y^{(1)} = \int_0^x \frac{x^2}{0+1}\,dx = \frac{x^3}{3}$$

$$y^{(2)} = \int_0^x \frac{x^2}{(\frac{1}{3}x^3)^2+1}\,dx$$

$$= \int_0^x \frac{x^2}{\frac{1}{9}x^6+1}\,dx$$

Putting $\frac{1}{3}x^3 = t$

$$\frac{3}{3}\cancel{x^2}dx = dt$$

$$\cancel{x^2dx = dt}$$

$$x \Rightarrow 0 \Rightarrow t=0$$

$$x = (3t)^{1/3}$$

$$y^{(2)} = \int_0^{(3t)^{1/3}} \frac{dt}{t^2+1} = \tan^{-1} t = \tan^{-1}(\tfrac{1}{3}x^3)$$

$$y^{(2)} = \frac{1}{3}x^3 - \frac{1}{81}x^9 + \ldots\ldots$$

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