* Numerical Solution of Ordinary Differential Equations:-

In general closed-form solutio and recourse must be made to numerical methods for solving such differential equatio

Various numerical methods for the solution of ordinary differential equations. we consid the general first order differential equation

$$\frac{dy}{dx} = f(x, y) \quad \text{---} (1)$$

with the initial condition $f(x_0) = y_0$ — ②

In general we applied to the solution of syste-s of first order differential equations and will yield the solution in one of the two form

(i) A series for y in terms of powers of x, from which the value y canbe obtained by direct substitution.

(ii) A set of tabulated values of x and y

Then the methods of Taylor and Picard belong to case (i), whereas Euler, Runge-kutta Adams-Bashforth etc. belong to case (ii).

* Solution by Taylor's Series:-

We consider the first order differential equation,

$$\frac{dy}{dx} = y' = f(x, y)$$

with the initial condition

$$f(x_0) = y_0$$

If $y(x)$ is the exact solution of differential Equation (1) then the Taylor's series-for $y(x)$ around $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0) y_0' + \frac{(x-x_0)^2}{\angle 2} y_0'' \ldots \quad (2)$$

If the values of $y_0', y_0'' \ldots$ are known then Equation (1) gives a power series for y.

Using the formula for total derivatives we can write.

$$y'' = f' = f(x) + y' f_y = f_x + f f_y$$

where the suffixes denote partial derivatives with respect to the variable concerned.

Similarly, we obtain

$$y''' = f'' = f_{xx} + f_{xy} \cdot f + f(f_{yx} + f_{yy} f) + f_y (f_x + f_y f)$$

$$y''' = f_{xx} + 2 f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2$$

and other higher derivatives of y.

The method can easily be extended to simultaneous and higher-order differential equations.

Exp. From the Taylor's series for $y(x)$, find $y(0.1)$ correct to four decimal places if $y(x)$ satisfies

$$y' = x - y^2 \quad \text{and} \quad y(0) = 1 \Rightarrow x_0 = 0$$

Solution The Taylor's series for $y(x)$

$$y(x) = 1 + x y_0' + \frac{x^2}{2} y_0'' + \frac{x^3}{6} y_0''' + \ldots$$

The derivatives $y_0', y_0'', y_0''' \ldots$ etc are obtained from

$$y'(x) = x - y^2 \qquad y_0' = 0 - [y(0)]^2$$

Scanned with CamScanner

$$\Rightarrow y'_0 = -1$$

$$y''(x) = 1 - 2y.y' \Rightarrow y''(0) = 1 - 2y(0)\,y'(0)$$
$$= 1 - 2(+1)(-1) = 3$$

$$y'''(x) = 0 - 2y.y'' - 2y'^2 \Rightarrow y'''(0) = -2(1)(3) - 2(-1)^2$$
$$y'''(0) = -6 - 2 = -8$$

$$y^{iv}(x) = -2y.y''' - 6y'y'' \Rightarrow y^{iv}(0) = (-2)(1)(-8) - 6(-1)(3)$$
$$y^{iv}(0) = 16 + 18 = 34$$

$$y^{v}(x) = -2yy^{iv} - 8y'y''' - 6y''^2 \Rightarrow y^{v}(0) = -2(1)(34) - 8(-1)(-8)$$
$$- 6 \times 9$$
$$y^{v}(0) = -68 - 64 - 54 = -186$$

Using these values in the Taylor series

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 - \frac{31}{20}x^5 \ldots$$

Putting $x = 0.1$

$$y(0.1) = 1 - 0.1 + \frac{3}{2}(0.1)^2 - \frac{4}{3}(0.1)^3 + \frac{17}{12}(0.1)^4 - \frac{31}{20}(0.1)^5 \ldots$$

$$y(0.1) = 0.90 + .015 - 0.00133 + 0.00014167 - 0.0000155$$
$$= 0.91379$$
$$y(0.1) = 0.9138$$

Ans