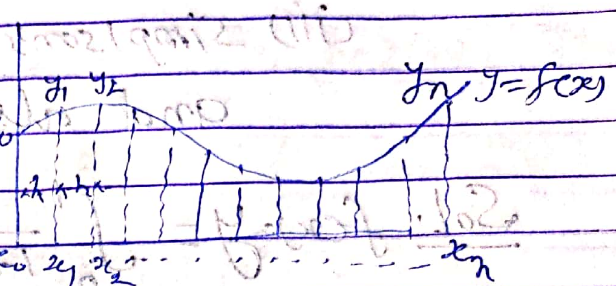
Numerical Integration:

The area bounded by the curve $f(x)$ and x axis between limit a & b is derived by $I = \int_a^b f(x)\,dx$.

Divide the interval (a,b) into n equal interval with length h (step size)

i.e. $(a,b) = \{a = x_0, x_1, x_2, \ldots, b = x_n\}$



$a_0 = x_0$

$x_1 = x_0 + h$

$x_2 = x_1 + h$

$\ldots$

$\vdots$

$x_n = x_{n-1} + h$

$n = \frac{b-a}{h}$ or $h = \frac{b-a}{n}$

→ Trapezoidal Rule is applicable on any no. of interval (i.e. even or odd interval number ($n = 1, 2, 3, 4, \ldots$)

$$I = \int_a^b f(x)\,dx = \int_a^b y\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \cdots y_{n-1}) + y_n\right]$$

→ Simpson's $\frac{1}{3}$ Rule is applicable on even number of interval ($n = 2, 4, 6, 8, \ldots$) i.e. total no. of even.

$$I = \int_a^b f(x)\,dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + \cdots) + y_n\right]$$

→ Simpson's $\frac{3}{8}$ Rule is applicable ~~if~~ total no. of interval is multiple of 3 i.e. ($n = 3, 6, 9, \ldots$)

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[y_0 + 3(y_1 + y_2 + y_4 + y_5 + y_7 \cdots) + 2(y_3 + y_6 + y_9 + \cdots) + y_n\right]$$

Exp. Evaluate $y = \int_0^1 \frac{1}{1+x^2} dx$ using

(i) Trapezoidal Rule

(ii) Simpson's $\frac{1}{3}$ Rule

(iii) Simpson's $\frac{3}{8}$ Rule

and also find value of π (PI)

Sol: $f(x) = y = \int_0^1 \frac{1}{1+x^2} dx$

$a = 0$ & $b = 1$ and Let $n = 6$

If n is not given then we will consider $n = 6$

then

$$h = \frac{1-0}{6} = 1/6$$

$$x_0 = 0 \Rightarrow y_0 = \frac{1}{1+x_0^2} = 1$$

$$x_1 = 0 + 1/6 = 1/6 \Rightarrow y_1 = \frac{1}{1+(\frac{1}{6})^2} = \frac{36}{37}$$

$$x_2 = \frac{1}{6} + \frac{1}{6} = 2/6 \Rightarrow y_2 = \frac{1}{1+\frac{4}{36}} = \frac{36}{40} = 0.9$$

$$x_3 = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} \Rightarrow y_3 = \frac{1}{1+\frac{9}{36}} = \frac{36}{45} = 0.8$$

$$x_4 = \frac{3}{6} + \frac{1}{6} = \frac{4}{6} \Rightarrow y_4 = \frac{1}{1+\frac{16}{36}} = \frac{36}{52} = \frac{9}{13}$$

$$x_5 = \frac{4}{6} + \frac{1}{6} = \frac{5}{6} \Rightarrow y_5 = \frac{1}{1+\frac{25}{36}} = \frac{36}{61}$$

$$x_6 = \frac{5}{6} + \frac{1}{6} = \frac{6}{6} \Rightarrow y_6 = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

(I) using Trapezoidal Rule.

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{1}{6\times 2}\left[\frac{37}{37} + 2\times\left(\frac{36}{37} + 0.9 + 0.8 + \frac{9}{13} + \frac{36}{61}\right) + 0.5\right]$$

$$= \frac{1}{12}\left[1.5 + 2(0.973 + 0.9 + 0.8 + 0.692 + 0.59)\right]$$

$$= \frac{1}{12}(1.5 + 2\times 3.955) = \frac{9.41}{12} = 0.78423 \quad \text{--- (1)}$$

(ii) By using Simpson's $\frac{1}{3}$ Rule.

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[y_0 + 4(y_1+y_3+y_5) + 2(y_2+y_4) + y_6\right]$$

$$= \frac{1}{6\times 3}\left[1+0.5 + 4\left(\frac{36}{37} + 0.8 + \frac{36}{61}\right) + 2\left(0.9 + \frac{9}{13}\right)\right]$$

$$= \frac{1}{18}\left[1.5 + 4(0.973 + 0.8 + 0.59) + 2(0.9 + 0.692)\right]$$

$$= \frac{1}{18}\left[1.5 + 4\times 2.363 + 2\times 1.592\right]$$

$$= \frac{1}{18}\times 14.136 = 0.785396 \quad \text{—— (2)}$$

(iii) By using Simpson's $\frac{3}{8}$ Rule

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{3h}{8}\left[y_0 + 3(y_1+y_2+y_4+y_5) + 2(y_3) + y_6\right]$$

$$= \frac{3\times 1}{8\times 6}\left[1 + 3(0.973 + 0.9 + 0.692 + 0.59) + 2(0.8) + 0.5\right]$$

$$= \frac{1}{16}\left[1.5 + 3\times 3.155 + 2\times 0.8\right]$$

$$= \frac{1}{16}\times 12.565 = 0.785394 \quad \text{—— (3)}$$

By Direct method

$$I = \int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \left[\tan^{-1} x\right]_0^1$$

$$= \left[\tan^{-1} 1 - \tan^{-1} 0\right]$$

$$= \tan^{-1}\tan \pi/4 - 0 = \pi/4 \quad \text{—— (4)}$$

Scanned with CamScanner

From (1) & (4) by trapezoidal rule

$$\frac{\pi}{4} = 0.78423$$

$$\pi = 4 \times 0.78423$$

$$\pi = 3.13692$$

From (2) & (4) by Simpson's $\frac{1}{3}$ Rule

$$\frac{\pi}{4} = 0.785396$$

$$\pi = 4 \times 0.785396$$

$$= 3.141584$$

From (3) and 4 by Simpson's $\frac{3}{8}$ Rule

$$\frac{\pi}{4} = 0.785394$$

$$\pi = 4 \times 0.785394$$

$$\pi = 3.141576$$