Abstract Algebra

NORMAL EXTENSIONS

Definition:- Let K be an algebraic extension of a field F. Then K is said to be normal extension of F, if the splitting field of the minimal polynomial $f(x) \in F[x]$ for each element of K is contained in K.

Some theorems on Normal Extensions

Theorem 1 :- A finite algebraic extension of K of a field F is normal over F if and only if K is the splitting field of some polynomial over F.

Proof :- Let $K = F(\alpha_1, \alpha_2, ---, \alpha_n)$ be a finite algebraic extension of F, where each α_i is algebraic over F. Let $f_1(x), f_2(x), --, f_n(x)$ be the minimal polynomials of $\alpha_1, \alpha_2, -- \alpha_n$ over F respectively.

Suppose that K is normal over F, then the splitting field of each $f_1(x), f_2(x), ----, f_n(x)$ is contained in K. Thus K is a splitting field of a polynomial $f(x) = f_1(x) f_2(x) --- f_n(x)$ over F.

Conversely, suppose K is a splitting field of some polynomial $f(x) \in F[x]$ and let $\alpha_1, \alpha_2, ---, \alpha_m$ be the roots of $f(x)$ in K, then

$$K = F(\alpha_1, \alpha_2, --, \alpha_m)$$

Now we shall show that K is normal over F. For K to be normal, we shall show that the splitting field of the minimal polynomial for each element of K is contained in K.

Let $\alpha \in K$ be an arbitrary element and let $p(x)$ be the minimal polynomial of α over F. Let if possible $p(x)$ do not have all of its roots in K.

Let β be a root of $p(x)$ which is not in K.

Then by lemma $F(\alpha)$ is isomorphic to $F(\beta)$ such that α is mapped on β and each element of F remains fixed under that isomorphism.

Since $\alpha \in K$, then the field K is the splitting field of $f(x)$ over $F(\alpha)$. Also $K(\beta)$ is generated by the root of $f(x)$ over $F(\beta)$, this $K(\alpha)$ is the splitting field of $f(x)$ over $F(\beta)$.

Therefore an isomorphism of $F(\alpha)$ onto $F(\beta)$, is extended to an isomorphism of K onto $K(\beta)$ such that each element of F remains fixed under this isomorphism. Hence K and $K(\beta)$ will have same degree over F which gives that $K(\beta)$ can not be a proper extension of K which gives a contradiction because $K \subseteq K(\beta)$.

Hence K is a normal extension of F.

<u>Theorem 2</u> :- Let K be a normal extension of a field F and L is an intermediate field so that $F \subset L < K$, then K is also a normal extension of L.

<u>Proof</u> :- K is a normal extension of F and L is a field such that $F \subset L \subset K$. Now we shall show that K is a

normal extension of L. We actually show that K is the splitting of a minimal polynomial for every element of K over L is Contained in K.

Let $\alpha \in K$ be any element and let $f(x)$ and $g(x)$ be the minimal polynomial for α over F and L respectively.

Since $f(x) \in F[x]$, then $f(x) \in L[x]$. Also $g(x)$ is the minimal polynomial of α over L, then

$$g(x) \mid f(x).$$

This shows that every root of $g(x)$ is the root of $f(x)$. But every root of $f(x)$ is in K, so that every root of $g(x) \in L[x]$ is in K. Hence K is normal extension of L.