Abstract Algebra

NORMAL SUBGROUPS

Let H be a subgroup of an abelian group G, then $Hx = xH$ $\forall x \in G$

Sometimes, it is possible that G is a non-abelian group possesses a subgroup H such that $Hx = xH$, $\forall x \in G$. Such subgroups whose left and right cosets coincide are called normal subgroup.

Definition :- A subgroup H of a group G is said to be a normal subgroup of G if for every $x \in G$ and for every $h \in H$, $xhx^{-1} \in H$.

Remarks

(i) Galois had given special attention first times, on such types of groups.

(ii) A normal subgroup is also called an invarient subgroup or a self conjugate subgroup or a normal divisor of the group.

(iii) Every group G possesses at least two normal subgroups, namely G itself and the subgroup consisting of the identity element e alone i.e. $\{e\}$. These are called improper normal subgroups of G.

(iv) From above definition we just conclude that H is a normal subgroup of G if and only if $xHx^{-1} \subseteq H \;\forall\, x \in G$.

SIMPLE GROUPS :-

A group $G \neq \{e\}$ is Known as simple group of it has no proper normal subgroup.

HAMILTONIAM GROUPS :-

A non-abelian group, each of whose subgroup is normal, is said to be Hamiltonian group.

* Every subgroup of an abelian group is normal.

Example :- If G is a group and H is a subgroup of index 2 in G, show that H is a normal subgroup of G.

Solution :- since index of H in G is 2, there are two distinct left (right) Cosets of H is G. Also we have $H = eH = He$ so H is itself a left as well as right Coset of H in G.

Let $a \in G$. If $a \in H$ then
$$aH = Ha = H.$$

Again for $a \in G$, $a \notin H$ the left

Theorem :- A subgroup H of a group G is normal iff $xHx^{-1} = H \quad \forall x \in G$.

Proof :- Let us first suppose

$$xHx^{-1} = H \quad \forall x \in G$$

Then $xHx^{-1} \subseteq H, \ \forall x \in G$

$\Rightarrow$ H is a normal subgroup of G.

Conversely, let H be a normal subgroup of G. then

$$xHx^{-1} \subseteq H \quad \forall h \in H \text{ and for all } x \in G.$$

$$\Rightarrow xHx^{-1} \subseteq H \quad \forall x \in G.$$

Also, let h be any element of H, then for every $x \in G$, we have

$$h = ehe = xx^{-1}hxx^{-1} = x(x^{-1}hx)x^{-1} \in xHx^{-1}$$

since $x^{-1}hx = x^{-1}h(x^{-1})^{-1} \in H$, ~~as~~ $x^{-1} \in G$ and H being normal.

Therefore,

$$H \subset xHx^{-1}, \ \forall x \in G.$$

Now from 1 and 2, we conclude that

$$xHx^{-1} = H, \ \forall x \in G$$

$$\Rightarrow (xHx^{-1})x = Hx, \ \forall x \in G$$

$$\Rightarrow (xH)x^{-1}x = Hx, \ \forall x \in G$$

$$\Rightarrow xH = Hx, \ \forall x \in G$$

$\Rightarrow$ Each left coset xH is the right coset Hx.

Cosets aH is different from H and likewise the right coset Ha is different from H. since there are only two distinct left (right) cosets of H in G, the decomposition of G into left cosets with respect to H cosist of H and aH for $a \in G$, $a \notin H$. therefore

$$G = H \cup aH$$

Also, decomposing G into right cosets with respect to H, we have

$$G = H \cup Ha.$$

Now the left cosets H and aH as well as the right cosets H and Ha have no element in common (since they are disjoint, it follows from the above relation). we must have

$$Ha = aH$$

Now, since a is arbitrary, therefore every left cosets of H is also a right coset of H.

Hence, H is a normal subgroup of G.