3) Equation $\{F(x,y,p)=0\}$ — (i) is solvable for x.

Solving Eq. $F(x,y,p)=0$ for x leads to

$$x = \varphi_i(y,y'), \quad i=1,2,\dots l \quad \text{—— (ii)}$$

where l is a number of solutions. Further we must solve each of the l equations.

Let $p(y) = y'(x)$ is new function of y alone.

A diff. of Eqn (ii) w.r.to x then yields, taking

$$\frac{dp(y)}{dx} = pp'(y) \text{ into account.}$$

$$1 = \varphi'_{iy}(y,p)p + \varphi'_{ip}(y,p)pp'(y)$$

Then $$p'(y) = \frac{1 - p\varphi'_{iy}(y,p)}{p\varphi'_{iy}(y,p)}, \quad i=1,2,3\dots l \quad \text{—— (iii)}$$

This diff. Eqn is completely analogous to Eqn (4)

$$y'(x) = f_i(x,y), \quad i=1,2,\dots l \quad \text{—— (iv)}$$

If an explicit function $p(y,c)$ is a general solution of eqn (iii), then substituting this function in Eqn (ii) one finds a general solution of Eqn (i) as an iverse function $x = \varphi_i(y, p_i(y,C))$. If an inverse function $y = y_i(p,C)$ is the general solution of Eqn (iii), then this solution together with Eqn (ii) give a general solution of Eqn (i) in the parametric form $y = y_i(p,C)$, $x = \varphi_i(y_i(p,C),p)$, where p is parameter

Exp Find the general solution of the diff. Eqn.

$$y'^3 - y - x = 0$$

Solution We have

$$y'^3 - y = x$$

or $x = p^3 - y$ —— (1)

$\because$ $p = y' = \frac{dy}{dx}$

on differentiating Eqn (1) w.r.to x

$$1 = 3p^2 \frac{dp}{dx} - \frac{dy}{dx}$$

$$1+p = 3p^2 \frac{dp}{dy} \cdot \frac{dy}{dx} \Rightarrow 1+p = 3p^3 \frac{dp}{dy}$$

$$p'(y) = \frac{1+p}{3p^3}$$

$$\frac{dp}{dy} = \frac{1+p}{3p^3}$$

$$dy = \frac{3p^3}{(1+p)} dp$$

$$dy = \frac{(3p^3+3-3)}{1+p} dp$$

$$= \frac{3(p^3+1)}{1+p} dp - \frac{3\,dp}{1+p}$$

$$dy = \frac{3\cancel{(1+p)}(p^2+1-p)}{\cancel{(1+p)}} dp - 3\frac{dp}{1+p}$$

$$dy = 3(p^2 - p + 1)dp - 3\frac{dp}{1+p}$$

Integrating both sides

$$\int dy = 3\left[\int p^2 dp - \int p\,dp + \int dp\right] - 3\int \frac{dp}{1+p} + c$$

$$y = 3\left[\frac{p^3}{3} - \frac{p^2}{2} + p\right] - 3\log(1+p) + c$$

$$y = p^3 - \frac{3}{2}p^2 + 3p - 3\log(1+p) + c,$$

$$x = p^3 - y, \text{ if } p \neq -1$$

where c is an arbitrary constant. At $p = -1$ we find the singular solution.

$$x = -1 + y.$$