* A sequence $\langle a_n \rangle$ is said to converge to a number l, if given any $\epsilon > 0$, there exists a positive integer m (depending on ϵ) such that

$$|a_n - l| < \epsilon \quad \forall\, n \geq m$$

The number l is called the limit of the sequence $\langle a_n \rangle$ and is written as

$$\lim_{n \to \infty} a_n = l \quad \text{or} \quad a_n \to l$$

Exp Show that $\lim_{n \to \infty} \frac{1}{n} = 0$

Sol. Let $\epsilon > 0$ be any number

If $a_n = \frac{1}{n}$ ~~then~~ & $l = 0$ then. $|a_n - l| < \epsilon$

$|a_n - 0| = |\frac{1}{n} - 0| = |\frac{1}{n}| = \frac{1}{n} < \epsilon$, for $n \geq$

Let m be a positive integer such that

$m > \frac{1}{\epsilon}$. Then

$|a_n - 0| < \epsilon \quad \forall\, n \geq m$

Hence $\lim_{n \to \infty} \frac{1}{n} = 0$.

Exp. Show that
$$\lim_{n\to\infty}\left(1+\frac{(-1)^n}{n}\right)=1$$

Sol. Let $\epsilon>0$ be any number Then

If $a_n = 1+\frac{(-1)^n}{n}$

we know

$|a_n - l| < \epsilon \quad \forall\ n\geq m \quad \left(\because \lim_{n\to\infty} a_n = l\right)$

$$|a_n-1| = \left|1+\frac{(-1)^n}{n}-1\right| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n} < \epsilon,$$

if $n > \frac{1}{\epsilon} \Rightarrow n\geq m$

let m be a positive Itegar $> \frac{1}{\epsilon}$ Then

$|a_n - l|<\epsilon \quad \forall\ n\geq m$

Hence $\lim_{n\to\infty}\left(1+\frac{(-1)^n}{n}\right)=1$

Exp. Prove that if $p>0$, then $\lim_{n\to\infty}\frac{1}{n^p}=0$

Sol:- let $\epsilon>0$ be given any number. Then

$$|a_n - 0| = \left|\frac{1}{n^p} - 0\right| = \left|\frac{1}{n^p}\right| = \frac{1}{n^p} < \epsilon \Rightarrow n^p > \frac{1}{\epsilon}$$

$\Rightarrow n > \left(\frac{1}{\epsilon}\right)^{1/p} \Rightarrow n\geq m$

Where m is a positive integer $m \geq \left(\frac{1}{\epsilon}\right)^{1/p}$

Thus $|a_n - 0| < \epsilon$ for $n\geq m$

$\Rightarrow \lim_{n\to\infty}\frac{1}{n^p}=0$, for $p>0$

* Necessary condition for convergence :- If a positive term series $\sum a_n$ is convergent then

$$\lim_{n\to\infty} a_n = 0$$

Proof Let

$$S_n = a_1 + a_2 + \cdots + a_n$$

Since $\sum a_n$ is given to be convergent then.

$$\lim_{n\to\infty} S_n = k \text{ (a finite quantity)}$$

Aslo

$$\lim_{n\to\infty} S_{n-1} = k$$

But

$$a_n = S_n - S_{n-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = 0$$

$$\lim_{n\to\infty} a_n = 0$$

Hence. The result.

* It is important to note that the converse of this result is not true.

Exp. Consider the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$$

Since the term go on descending.

Sol. Since

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$$

$$> \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots \frac{1}{\sqrt{n}}$$

$$\geq \frac{n}{\sqrt{n}} \quad \text{i.e. } \sqrt{n}$$

$$S_n = \sqrt{n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (\sqrt{n}) \to \infty$$

Thus the series is divergent even though

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}}\right) = 0$$

Hence $\lim_{n\to\infty} a_n = 0$ is necessary but not sufficient condition for convergent of Σa_n series.