Abstract Algebra

MODULES

Let R be any ring. We call it the "ring of scalars" and write its elements as lower case letter r, s, t ...

1. **Definition** Let R be an arbitrary rings, then a non-empty set M is said to be an R-module if M is an abelian group under addition and there exists $ra \in M$ for every $r \in R$ and $a \in M$ and subjected to the following axioms for all elements $r, s \in R$ and $a, b \in M$:

(i) $r(a+b) = ra + rb$

(ii) $(r+s)a = ra + sb$

(iii) $r(sa) = (rs)a$

Notes :-

(i) More explicitly, such a module M is a left module, because informing ra, the scalar r is written on the left of the module element a. Similarly we can define a right R-module.

2. **Definition** unital R-module

Let R be a ring with unity then an R-module M is said to be unital R-module if $1 \cdot a = a$ for all $a \in M$.

Notes :-

(i) If R is a field, then a unital R-module form a vector space over R.

(ii) We shall treat left R-module as module.

(iii) Every abelian group G is a unital module over the ring of integers.

(iv) If R is a ring and S is a subring of R, then S is an R-module (module over R)

SUBMODULE

Definition :- Let M be an R-module and N be a non-empty subset of M. Then N is said to be a submodule of M if :

(i) N is an additive subgroup of M

(ii) $ra \in N$ for $a \in N$ and $r \in R$

Among the submodules of M are M itself and the set (0) consisting of the zero element alone. Any submodule of M different from these two submodules M and (0) are called proper submodule and $M, (0)$ are called improper submodule or irreducible submodule.

Notes :-

(i) Intersection of the submodules of a Module M is a submodule.

(ii) Arbitrary intersection of submodules of a module M is a submodule.

LINEAR SUM OF SUBMODULES

Linear Sum of Submodules let A and B be two submodules of an R- module M, then the set

$$A+B = \{a+b : a \in A, b \in B\}$$

is called the linear sum of A and B.

Notes :-

(i) The linear sum of two submodules of an R- module M is also a submodule of M.

QUOTIENT MODULES

The construction of quotient modules is like that of quotient groups. let M be an R- module and N a submodule. Since N is obviously a normal subgroup of the additive group of M. Hence the quotient group M/N exists. let $a+N$ be a coset of N in M, and let $a \in M$. Thus we have

$$M/N = \{a+N : a \in M\} = \text{set of all cosets of } N \text{ in } M$$

Now we define $x(a+N) = xa+N$ for $x \in M$. It is trivally verified that this operation is well defined and this operation on M/N satisfies all the conditions of a operation of R. Thus M/N forms a module. This module is called quotient module of M by its submodule N.