System of linear Equations :-

(8) (a) **Consistent system of equations :-**

If a set of equations has at least one solution, it is said to be consistent, otherwise it is said to be inconsistent.

(b) **Homogeneous system :-**

The system of equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}\right] \text{— (I)}$$

in n unknowns $\{x_1, x_2, \ldots, x_n\}$ is called a homogeneous system of linear equations.

Since $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$ satisfies (I).

$\therefore$ The homogeneous system (I) is always consistent.

i.e. (I) is consistent whatever be the coefficients a_{ij}. The solution $(0, 0, \ldots, 0)$ is called a trivial or zero solution of (I). Any other solution is called a non-trivial solution.

The system (I) has a non-trivial solution if the rank of the Co-efficient matrix $A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix}$

is less than n.

The number of independent solutions of the homogeneous system (I) is $n-r$, where r is the rank of the Co-eff. matrix A.

Putting $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. the

system (I) can be written as

$$AX = O \quad \text{—} (I')$$

In order to solve (I) [or (I')] we transform A to its echelon form by applying elementary row operations.

(c) Non-homogeneous system of linear equations :-

The system of linear equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \text{—} (I)$$

in n unknowns $x_1, x_2, \ldots, x_n$ is called a non-homogeneous system of linear equations if not all b's are zero.

Putting $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$,

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

The set of equations (I) can be written in the matrix form as

$$AX = B \quad \text{——} \quad (I')$$

The system (I) which is the same as (I') is consistent if and only if rank A = rank $[AB]$

i.e. iff rank $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

$= \text{rank} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$