Linear Equation :-

The general form of a first order linear differential equation is as -

$$A(x)\frac{dy}{dx} + B(x)y + C(x) = 0$$

$$\frac{dy}{dx} + \frac{B(x)}{A(x)}y = -\frac{C(x)}{A(x)}$$

It can be put in the form -

or $$\frac{dy}{dx} + P(x)y = Q(x) \quad —— \quad (1)$$

If the second member is zero, $Q(x)=0$, the equation can be solved by separating the variables

$$\frac{dy}{dx} + P(x)y = 0 \quad —— \quad (2)$$

$$\frac{dy}{y} = -P(x)\,dx$$

$$\log(y) = -\int P(x)\,dx + c$$

$$y = c_1 \exp\left[-\int P(x)\,dx\right] \quad —— \quad (3)$$

where c_1 is an arbitrary constant.

In general, we multiply Eqⁿ(1) by $\exp[\int p(x)dx]$

then

$$\exp\left[\int p(x)dx\right]\left[\frac{dy}{dx} + p(x)y\right] = \exp\left[\int p(x)dx\right]Q(x) \quad \text{--- } (4)$$

The first member in (4) is now recognized as derivative of $\exp[\int p(x)dx]y$

$$\left(\exp\left[\int p(x)dx\right]y\right)' = \left(\exp\left[\int p(x)dx\right]\right)'y + \exp\left[\int p(x)dx\right]y'$$

$$= \exp\left[\int p(x)dx\right]p(x)y + \exp\left[\int p(x)dx\right]y'$$

and we can get the general solution by an integration of (4)

$$\exp\left[\int p(x)dx\right]y = \int \exp\left[\int p(x)dx\right]Q(x)dx + c$$

$$y = c\left[\exp\left(-\int p(x)dx\right)\right] + \exp\left[-\int p(x)dx\right]\cdot\int e^{\int p(x)dx}Q(x)dx$$

$$y = c\,e^{-\int p(x)dx} + e^{-\int p(x)dx}\cdot\int e^{\int p(x)dx}Q(x)dx$$

$$y = c\,e^{-\int p(x)dx} + e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}Q(x)dx\right]$$

Where c is an arbitrary constant

Exp. Solve the differential equation

$$xy' - 2y = 2x^4 \quad \text{---} (1)$$

Solution- In order to use the method of variation of parameters, it is necessary to know a solution of the corresponding homogeneous equation

$$xy' - 2y = 0$$

$$x\frac{dy}{dx} = 2y$$

$$\frac{dy}{y} = 2\frac{dx}{x}$$

Integrating

$$\int \frac{dy}{y} = 2\int \frac{dx}{x} + \log|c|$$

$$\log|y| = 2\log|x| + \log|c|$$

$$y = cx^2$$

Then we assume that the general solution of the original equation is of the form

$$y = A(x)\, x^2 \quad \text{---} (2)$$

Eq (2) differentiating w.r.to x

$$y' = 2x A(x) + x^2 A'(x)$$

Substituting y & y' in Eqn (1)

$$x[2x A(x) + x^2 A'(x)] - 2A(x)\, x^2 = 2x^4$$

$$A'(x) = 2x \Rightarrow A(x) = x^2 + 9$$

Finally, substituting for $A(x)$ in equation (2) gives the general solution of Eqn (1) in the form.

$$y = (x^2 + c)x^2$$

* Otherway of the solution of given diff. Eqn

$$x\frac{dy}{dx} - 2y = 2x^4$$

$$\frac{dy}{dx} - \frac{2}{x}y = 2x^3 \quad \text{to compare } \frac{dy}{dx} + p(x)y = Q(x)$$

$$p(x) = -\frac{2}{x}$$

$$Q(x) = 2x^3$$

$$\text{Here } I.F = e^{\int p(x)\,dx} = e^{\int -\frac{2}{x}dx}$$

$$= e^{-2\int \frac{1}{x}dx}$$

$$I.F = e^{-2\log x} = \frac{1}{x^2}$$

Thus.

$$\frac{1}{x^2}y = \int 2x^3 \cdot \frac{1}{x^2}\,dx + c$$

$$\frac{1}{x^2}y = \int 2x\,dx + c$$

$$\frac{1}{x^2}y = \frac{2x^2}{2} + c$$

$$y = x^2(x^2 + c)$$