Linear Transformation

Exp. Show that the mapping $T : V(R^2) \to V(R^3)$ defined as $T(u,v) = (u+v, u-v, v)$ $\forall\ u, v \in V(R^2)$

Solution Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ and $\alpha, \beta \in R$

Given that mapping $T : V(R^2) \to V(R^3)$ as condition

$$T(u,v) = (u+v, u-v, v) \quad \text{---} \ (1) \qquad u, v \in V(R^2)$$

We know that Linear Transformation

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

$$\because T(\alpha u + \beta v) = T[\alpha(u_1, v_2) + \beta(v_1, v_2)]$$

$$= T[(\alpha u_1, \alpha u_2) + (\beta v_1, \beta v_2)]$$

$$= T[(\alpha u_1 + \beta v_1, \alpha u_2 + \beta v_2)]$$

$$= T(\underbrace{\alpha u_1 + \beta v_1}_{u}, \underbrace{\alpha u_2 + \beta v_2}_{v})$$

Using condition (1)

$$T(u,v) = (u+v, u-v, v)$$

$$\because T(\alpha u + \beta v) = (\alpha u_1 + \beta v_1 + \alpha u_2 + \beta v_2, \alpha u_1 + \beta v_1 - \alpha u_2 - \beta$$
$$\alpha u_2 + \beta v_2)$$

$$T(\alpha u+\beta v) = (\alpha(u_1+u_2)+\beta(v_1+v_2), \alpha(u_1-u_2)+\beta(v_1-v_2), \alpha u_2+\beta v_2) \quad \text{—— (ii)}$$

$$= [\alpha(u_1+u_2, u_1-u_2, u_2)+\beta(v_1+v_2, v_1-v_2, v_2)]$$

$$= \alpha T(u_1,u_2)+\beta T(v_1,v_2)$$

by condition (1)

$$= \alpha T(u)+\beta T(v)$$

∴ Hence $T(\alpha u+\beta v) = \alpha T(u)+\beta T(v)$

Then Transformation is linear.

Alternative method

$$\alpha T(u)+\beta T(v) = \alpha T(u_1,u_2)+\beta T(v_1,v_2)$$

$$= \alpha(u_1+u_2, u_1-u_2, u_2)+\beta(v_1+v_2, v_1-v_2, v_2)$$

by condition (1)

$$= (\alpha u_1+\alpha u_2, \alpha u_1-\alpha u_2, \alpha u_2)+(\beta v_1+\beta v_2, \beta v_1-\beta v_2, \beta v_2)$$

$$= [\alpha(u_1+u_2)+\beta(v_1+v_2), \alpha(u_1-u_2)+\beta(v_1-v_2), (\alpha u_2+\beta v_2)]$$

from (ii)

$$\alpha T(u)+\beta T(v) = T(\alpha u+\beta v)$$