Exp. Solve $\frac{d^2y}{dx^2}+a^2y = \sec ax$

Solution Given differential equation in symbol form

$$(D^2+a^2)y = \sec ax$$

(i) To find C.F from $(D^2+a^2)y=0$

The auxiliary Equation is

$$D^2+a^2=0$$

$$D^2=-a^2$$

$$D=\pm ia$$

$$C.F = c_1 \cos ax + c_2 \sin ax$$

(ii) To find P.I from $\sec ax$

$$P.I = \frac{1}{D^2+a^2}\sec ax$$

$$= \frac{1}{(D+ia)(D-ia)}\sec ax \quad \text{By factorizing}$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia}-\frac{1}{D+ia}\right]\sec ax$$

Resolving into partial fraction

$$= \frac{1}{2ia}\left[\frac{1}{D-ia}\sec ax - \frac{1}{D+ia}\sec ax\right]$$

$$\because \frac{1}{f(D-a)}X = e^{ax}\int X\cdot e^{-ax}$$

$$= \frac{1}{2ia}\left[e^{iax}\int e^{-iax}\sec ax\,dx - e^{-iax}\int e^{iax}\sec ax\right.$$

$$\because e^{-iax} = \cos ax - i\sin ax$$

$$\because e^{iax} = \cos ax + i\sin ax$$

$$P.I = \frac{1}{2ia}\left[e^{iax}\int \frac{\cos ax - i\sin ax}{\cos ax}dx - e^{-iax}\int \frac{\cos ax + i\sin ax}{\cos ax}dx\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\left(\int dx - i\int \tan ax\, dx\right) - e^{-iax}\left(\int dx + i\int \tan ax\, dx\right)\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\left(x - \frac{i}{a}(-\log\cos ax)\right) - e^{-iax}\left(x + \frac{i}{a}(-\log\cos ax)\right)\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\left(x + \frac{i}{a}\log\cos ax\right) - e^{-iax}\left(x - \frac{i}{a}\log\cos ax\right)\right]$$

$$= \frac{x}{a}\left(\frac{e^{iax} - e^{-iax}}{2i}\right) + \frac{1}{a^2}\log\cos ax\left(\frac{e^{iax} + e^{-iax}}{2}\right)$$

$$P.I = \frac{x}{a}\cdot\sin ax + \frac{1}{a^2}(\log\cos ax)\cdot(\cos ax)$$

The general solution is

$$y = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{\cos ax\log\cos ax}{a^2}$$