Indeterminate form $(\infty - \infty)$

Exp. Evaluate $\lim\limits_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$

Sol. This is of the form $(\infty - \infty)$

$\therefore$ Then given limit $\lim\limits_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2(\sin^2 x)} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 - x^2}{x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2}$$

$$\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \lim_{x\to 0} \frac{\left(x^2 + \frac{x^6}{3!\,3!} + \left(\frac{x^5}{5!}\right)^2 + \cdots - 2\cdot x\cdot\frac{x^3}{3!} + \frac{2x\cdot x^5}{5!} \cdots \cdots\right) - x^2}{x^2\left[x^2 + \left(\frac{x^3}{3!}\right)^2 + \left(\frac{x^5}{5!}\right)^2 + \cdots - \frac{2x\cdot x^3}{3!} + \frac{2x\cdot x^5}{5!} \cdots\right]}$$

$$= \lim_{x\to 0} \frac{\left(-\frac{2x^4}{6} + x^6\left(\frac{1}{36} + \frac{2}{120}\right) - \cdots\right]}{\left(x^4 - \frac{2}{6}x^6 + \cdots\right)}$$

$$= \lim_{x\to 0} \frac{x^4\left[-\frac{2}{6} + x^2\left(\frac{1}{36} + \frac{1}{60}\right) + \cdots\right]}{x^4\left[1 - \frac{1}{3}x^2 + \cdots\right]}$$

$$= \lim_{x\to 0} \frac{\left(\frac{1}{3} - x^2\left(\frac{1}{36} + \frac{1}{60}\right) + \cdots\right]}{1 - \frac{1}{3}x^2 \cdots}$$

$$= \frac{1}{3}$$

$$\therefore \lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) = \frac{1}{3}$$

Indeterminate form $[\infty - \infty]$

Exp. Evaluate $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$

Sol. This is of the form $(\infty - \infty)$

$\therefore$ Then given limit $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^2 (\sin^2 x)} \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 - x^2}{x^2 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2}$$

$\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$$= \lim_{x \to 0} \frac{\left(\cancel{x^2} + \frac{x^6}{3! \cdot 3!} + \left(\frac{x^5}{5!}\right)^2 + \cdots - \frac{2 \cdot x \cdot x^3}{3!} + \frac{2x \cdot x^5}{5!} \cdots \cdots \right) - \cancel{x^2}}{x^2 \left[x^2 + \left(\frac{x^3}{3!}\right)^2 + \left(\frac{x^5}{5!}\right)^2 + \cdots - \frac{2x \cdot x^3}{3!} + \frac{2x \cdot x^5}{5!} \cdots \right]}$$

$$= \lim_{x \to 0} \frac{\left(-\frac{2x^4}{6} + x^6\left(\frac{1}{36} + \frac{2}{120}\right) - \cdots \right)}{\left(x^4 - \frac{2}{6}x^6 + \cdots \right)}$$

$$= \lim_{x \to 0} \frac{x^4\left[-\frac{2}{6} + x^2\left(\frac{1}{36} + \frac{1}{60}\right) + \cdots \right]}{x^4\left[1 - \frac{1}{3}x^2 + \cdots \right]}$$

$$= \lim_{x \to 0} \frac{\left(\frac{1}{3} - x^2\left(\frac{1}{36} + \frac{1}{60}\right) + \cdots \right]}{1 - \frac{1}{3}x^2 - \cdots}$$

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$\therefore \lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right) = \frac{1}{3}$