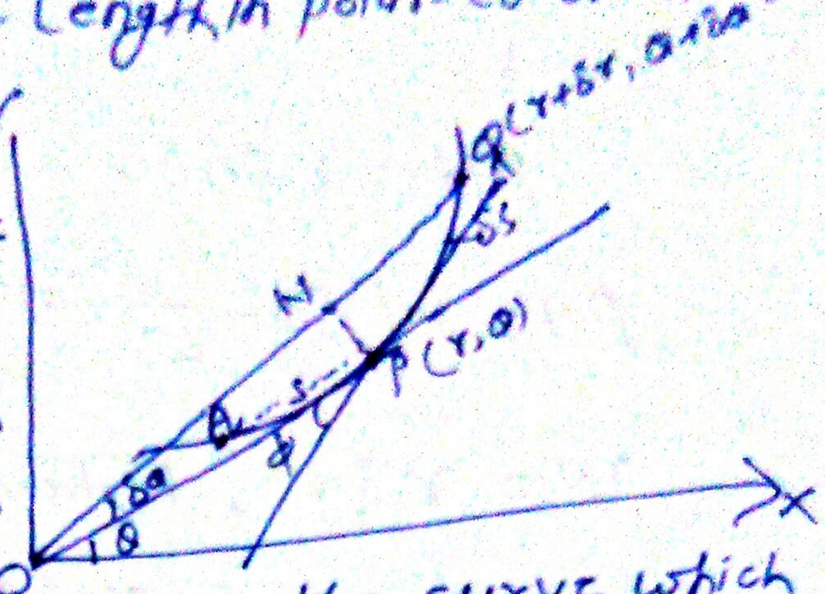
Polar-Co-ordinates

Derivatives of Arc-Length in polar-co-ordinates

Let O be the pole and OX the initial line let P be any point on curve and the length of Arc AP is S from any point A on curve.



let Q be any other point on the curve which is very to P. Then the Co-ordinates of Q will be $(r+\delta r, \theta+\delta\theta)$.

Arc $AQ = s+\delta s \Rightarrow$ arc $PQ = \delta s$

PH is $\perp^{lar}$ on OQ from $P(r,\theta)$ point on curve.

Now, in ΔPNQ $\quad OP = r$ and $OQ = r+\delta r$

~~$NQ = OQ - ON$~~

~~$NQ = \delta r$~~

$\because OP = r$

$$\sin\delta\theta = \frac{PN}{OP}$$

$$PN = OP\sin\delta\theta$$

$$PN = r\sin\delta\theta \quad \text{—— (I)}$$

$$\cos\delta\theta = \frac{ON}{OP}$$

$$ON = r\cos\delta\theta$$

$$NQ = OQ - ON = r+\delta r - r\cos\delta\theta$$

$$= \delta r + r(1-\cos\delta\theta)$$

$$= \delta r + r\cdot 2\cdot\sin^2\frac{\delta\theta}{2} \quad \text{—— (II)}$$

Now, if Q coincides with P i.e. if $\theta \to 0$ then in that case, we can write (i) & (ii) as follows

$$PN = r \sin \delta\theta$$

$$PN = r\left[\delta\theta - \frac{1}{\lfloor 3}(\delta\theta)^3 + \dots\right]$$ As Expansion $\sin\theta = \theta - \frac{1}{\lfloor 3}\theta^3 \dots$

$PN = r\delta\theta$, higher order will be neglect.

$$HQ = \delta r + 2r\left[\sin \frac{\delta\theta}{2}\right]^2$$

$$= \delta r + 2r\left[\left(\frac{\delta\theta}{2}\right) - \left(\frac{1}{\lfloor 3}\left(\frac{\delta\theta}{2}\right)^3\right) + \dots\right]^2$$

$$= \delta r + 2r\left[\left(\frac{\delta\theta}{2}\right)^2 - \frac{1}{\lfloor 3}\left(\frac{\delta\theta}{2}\right)^4 + \dots\right]$$

$HQ = \delta r$, up to first order of infinitesimal.

$PH = r\delta\theta$

$HQ = \delta r$

Also $\frac{\text{Chord } PQ}{\text{arc } PQ} \to 1$ when $\delta\theta \to 0$

$$\text{Chord } PQ = \text{arc } PQ = \delta s$$

In right ΔPHQ

$$PH = r\delta\theta,\ HQ = \delta r \ \&\ PQ = \delta s$$

$$(PQ)^2 = (PH)^2 + (HQ)^2$$

$$(\delta s)^2 = (r\delta\theta)^2 + (\delta r)^2$$

$$\Rightarrow \left(\frac{\delta s}{\delta\theta}\right)^2 = r^2 + \left(\frac{\delta r}{\delta\theta}\right)^2$$ Taking limit

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 \Rightarrow \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$