## Leibnitz's Theorem :- If $u$ and $v$ are two functions of $x$ such that their $n^{th}$ derivatives exist, then the $n^{th}$ derivative of their product is given by:-

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots {}^nC_r u_{n-r} v_r + \cdots u v_n.$$

where $u_r$ & $v_r$ represent $r^{th}$ derivatives of $u$ & $v$ respectively.

**Proof** :- By method of induction

Let $y = uv$ —— (1)

where $u, v$ are function of $x$

Eq^n (1) directly differentiating successively, we get

$$y_1 = u_1 v + v_1 u$$

$$y_2 = (u_2 v + u_1 v_1) + (v_2 u + v_1 u_1)$$

$$y_2 = u_2 v + 2 u_1 v_1 + u v_2$$

$$y_2 = u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2 \qquad (\because {}^2C_2 = 1)$$

$$y_3 = u_3 v + u_2 v_1 + 2(u_1 v_2 + u_2 v_1) + v_3 u + u_1 v_2$$

$$= u_3 v + 3 u_2 v_1 + 3 u_1 v_2 + v_3 u$$

$$y_3 = u_3 v + {}^3C_1 u_2 v_1 + {}^3C_2 u_1 v_2 + {}^3C_3 v_3 u \qquad \left(\begin{array}{l}\because {}^3C_1 = 3 \\ \because {}^3C_2 = 3\end{array}\right)$$

Thus, this theorem is true for $n = 1, 2, 3 \cdots$.
According to the law of induction, we assume that this theorem is true for $n = m$ i.e we shall get the same formal expression for $y_m$ as-

$$y_m = u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \cdots + {}^mC_{r-1} u_{m-r+1} v_{r-1} + {}^mC_r u_{m-r} v_r + \cdots + {}^mC_{m-1} v_{m-1} u_1 + {}^mC_m u v_m \quad \text{---} \ (2)$$

Differentiating once, we get

$$y_{m+1} = u_{m+1} v + u_m v_1 + {}^mC_1 (u_m v_1 + u_{m-1} v_2) + {}^mC_2 (u_{m-1} v_2 + u_{m-2} v_3) + {}^mC_{r-1} (u_{m-r+2} v_{r-1} + u_{m-r+1} v_r) + {}^mC_r (u_{m-r+1} v_r + u_{m-r} v_{r+1}) + {}^mC_{m-1} (u_2 v_{m-1} + u_1 v_m) + {}^mC_m (u_1 v_m + u v_{m+1})$$

$$y_{m+1} = u_{m+1} v + u_m v_1 ({}^mC_0 + {}^mC_1) + u_{m-1} v_2 ({}^mC_1 + {}^mC_2) + \cdots u_{m-r+1} v_r ({}^mC_{r-1} + {}^mC_r) + \cdots u_1 v_m ({}^mC_{m-1} + {}^mC_m) + {}^mC_m u v_{m+1}$$

We have $\quad {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$

Putting $r = 1, 3, 3 \cdots$

$${}^mC_0 + {}^mC_1 = {}^{m+1}C_1 \, ; \quad {}^mC_1 + {}^mC_2 = {}^{m+1}C_2 \cdots \text{etc}$$

Now

$$\Rightarrow y_{m+1} = u_{m+1} v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + \cdots + {}^{m+1}C_r u_{m-r+1} v_r + {}^{m+1}C_{m+1} u v_{m+1}$$

This theorem is also true for the next higher integer $n = m+1$. Hence this theorem is true for every value of $n$.