Lagrange's Mean Value Theorem

Q. State & prove Lagrange's mean value theorem

Statement:- If a function $f(x)$ be defined such that

(i) $f(x)$ is continuous at every point of the closed interval $[a, b]$

(ii) $f'(x)$ exists at every point of the open interval $]a, b[$,

then there exists at ~~every point of the open~~ least one value of x, say c such that

$$\frac{f(b) - f(a)}{b-a} = f'(c) \text{ where } a < c < b$$

Proof:- Let us define a function $\phi(x)$ such that

$$\phi(x) = f(x) + Ax \quad \text{——— (1)}$$

where A is constant to be chosen so that

$$\phi(a) = \phi(b) \quad \text{——— (2)}$$

Putting $x = a$ & $x = b$ in (1), we get

$$\phi(a) = f(a) + Aa$$

$$\& \ \phi(b) = f(b) + Ab$$

Substituting these values in (2), we get

$$f(a) + Aa = f(b) + Ab$$

or, $f(b) - f(a) = -A(b-a)$

or, $$-A = \frac{f(b) - f(a)}{b-a} \quad \text{—— (3)}$$

It is given that $f(x)$ is continuous in $[a, b]$ and Ax is also continuous. Therefore $f(x) + Ax$ i.e. $\phi(x)$ must be continuous in $[a, b]$

Diff. (1) w.r.t. x, we get

$$\phi'(x) = f'(x) + A \text{ ——— } (4)$$

It is given that $f'(x)$ exists in $]a, b[$, therefore from (4) $\phi'(x)$ must exist in $]a, b[$

Also from (2),

$$\phi(a) = \phi(b)$$

Thus we find that $\phi(x)$ satisfies all the conditions of Rolle's theorem.

Hence by the theorem there exists at least one value of x, say c such that

$$\phi'(c) = 0 \text{ —— } (5) \text{ where } a < c < b$$

from (4), we get

$$\phi'(c) = f'(c) + A$$

Using (5),

$$-A = f'(c)$$

or, $$\frac{f(b) - f(a)}{b - a} = f'(c) \quad [\text{from } (3)]$$