Lagrange's Equation:-

An equation of the form

$y = x\varphi(y') + f(y')$ is known as a Lagrange' equation. In this case Eqn (1)

$$p'(x) = \frac{p - \varphi'_{ix}(x,p)}{\varphi'_{ip}(x,p)} \quad \text{—— (1)} \quad i = 1, 3, \cdots l$$

becomes linear with respect to the inverse function $x(p)$ then-

$$\frac{dx}{dp} - x\frac{\varphi(p)}{p - \varphi(p)} = \frac{f'(p)}{p - \varphi(p)} \quad \text{—— (II)}$$

When $p - \varphi(p) \neq 0$. A solution of eqn(II) can be found by method of variation of parameter. Hence a general solution of the Lagrange equation can be always written in the parametric form

$$x = x(p, C), \quad y = x\varphi(p) + f(p)$$

Where $p\left(=\frac{dy}{dx}\right)$ is parameter and C is an arbitrary constant.

In those points p_i^* where $p_i^* - \varphi(p_i^*) = 0$, we have the singular solutions as the straight lines $y = x\varphi(p_i^*) + f(p_i^*)$

where $i = 1, 2, 3, \cdots k$ and k is a number of the singular solutions.

Exp. Find the general solution of the Lagrange's equation

$$xy'^2 - y' - y = 0$$

Solution Given that Lagrange's equation

$$xy'^2 - y' - y = 0$$

$$y = xy'^2 - y'$$

or $$y = xp^2 - p \quad \text{—— (1)}$$

where $p = \frac{dy}{dx} = y'$

on diff. w.r.to x Eqn (1) we get

$$\frac{dy}{dx} = x \cdot 2p\frac{dp}{dx} + p^2 - \frac{dp}{dx}$$

$$p = p^2 + \frac{dp}{dx}(2xp - 1)$$

$$p(1-p) = \frac{dp}{dx}(2xp-1)$$

$$\frac{dx}{dp} = \frac{2xp-1}{p(p-1)}$$

$$\frac{dx}{dp} = \frac{1}{p(p-1)} - \frac{2x}{p-1}$$

$$\frac{dx}{dp} + \frac{2x}{p-1} = \frac{1}{p(p-1)}$$

This is in linear diff. eqn form if $p \neq 0$ & $p \neq 1$

Then.

$$I.F = e^{\int \frac{2}{p-1}dp} = e^{2\log(p-1)} = (p-1)^2$$

Thus general solution is

$$x \cdot I.F = \int Q(p) \cdot I.F\, dp + c$$

$$x\cdot(p-1)^2 = \int \frac{1}{p(p-1)}\cdot(p-1)^2\,dp + c$$

$$x(p-1)^2 = \int \frac{p-1}{p}\,dp + c$$

$$= \int\left(1-\frac{1}{p}\right)dp + c$$

$$x\cdot(p-1)^2 = p - \log|p| + c$$

$$x = \frac{c+p-\log|p|}{(p-1)^2}$$

$y = xp^2 - p$ is general solution of the given equation.

*. If $p=0$ or $p=1$ we find two singular solutions $y=0$ and $y=x-1$, respectively