Matrices

Kinds of matrices

Symmetric and skew symmetric matrices :-

(i) Prove that the product of any matrix with its transpose is a symmetric matrix.

Proof :- Let A be an $m \times n$ matrix.
Then its transpose A' is an $n \times m$ matrix.

$\because$ no. of columns of $A = n =$ no. of rows of A'

$\therefore$ The product AA' is defined.

Now. $(AA')' = (A')' \cdot A' = AA'$

i.e. $(AA')' = AA'$.

This means that AA' is symmetric.

(ii) Show that each diagonal element of a skew-symmetric matrix is zero.

Proof :- We find note that a skew-sym. matrix is a square matrix.

Let $A = (a_{ij})_{n,n}$ be a skew-symmetric matrix.

Now, the transpose of A is $A' = (a'_{ji})_{n,n}$
where $a'_{ji} = a_{ij}$ ——— (1)

since A is skew symmetric.

$\therefore A' = -A$

$\therefore (j,i)^{th}$ element of $A' = -\{(j,i)^{th}$ element of $A\}$

i.e. $a'_{ji} = -a_{ji}$

$\therefore a_{ij} = -a_{ji}$ by (1)

Putting $i = j$, we get $a_{ii} = -a_{ii}$

or, $2a_{ii} = 0$

or, $a_{ij} = 0$

But a_{ii} is the diagonal element in the i^{th} row and the i^{th} Column of A. Hence each diagonal element of A is zero.

Imp (iii) Show that any square matrix can be expressed uniquely as the sum of a symmetric and a skew symmetric matrix

Proof:- Let A be any square matrix and A' its transpose.

We can write.

$$A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A') \quad \text{——— (1)}$$

Now $(A+A')' = A' + (A')' = A' + A = A + A'$

and hence $\frac{1}{2}(A+A')$

and so $A+A'$ is a symmetric matrix.

Also $(A-A')' = A' - A = -(A-A')$

and so $A-A'$ and hence $\frac{1}{2}(A-A')$ is a skew symmetric matrix.

Hence (1) shows that A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

We shall now show that the representation (1) is unique.

If possible suppose A can be expressed also as

$$A = P + Q \quad \text{——— (2)}$$

where P is symmetric and Q is skew symmetric. But $P \neq \frac{1}{2}(A+A')$

and $Q = \frac{1}{2}(A-A')$

Then

$$P' = P \quad \text{and} \quad Q' = -Q$$

$$\therefore P' + Q' = P - Q$$

i.e. $A' = P - Q$ —— ③ [$\because$ By ②. $A' = P' + Q'$]

Then ② and ③ we get

$$P = \frac{1}{2}(A+A')$$

and $Q = \frac{1}{2}(A-A')$

This shows that the representation ② is not different from the representation ① (of A as the sum of a symmetric matrix and a skew symmetric matrix).

Hence the representation ① is unique.

(iv) Show that for any matrix A the matrices $A+A'$ and $A-A'$ are sym. and skew-sym. respectively.

Proof :- We shall first show that if A and B be matrices of the same type, then $(A \pm B)' = A' \pm B'$ and $(A')' = A$ where prime denotes transposition.

[For proof, see (vi) properties of transposition.

We now come to the main theorem.

We first note that in order that $A+A'$ and $A-A'$

are defined. A and A' must be of the same type.

But we note that if A is an $m \times n$ matrix, then A' is an $n \times m$ matrix. Hence A and A' will be of the same type if and only if $m = n$ i.e. iff A is a square matrix.

Hence the theorem given to be proved is faulty and it should read as follows :-

"Show that for any square matrix A, the matrices $A+A'$ and $A-A'$ are symmetric and skew - symmetric respectively. We shall prove the theorem in this corrected form.

From what has been proved above we have

$$(A+A')' = A' + (A')' = A' + A = A + A'$$

This shows that $A+A'$ is a symmetric matrix.

Again, $(A-A')' = A' - (A')' = A' - A = -(A-A')$

This shows that $A-A'$ is a skew symmetric matrix

This completes the proof: