Jordan - Holder Theorem

Let G be a finite group with two compositions

$$G, H_1, H_2, ----, H_n = \{e\} \quad \text{—— (1)}$$

$$\text{and } G, K_1, K_2, ----, K_m = \{e\} \quad \text{—— (2)}$$

Then $n = m$ and the two corresponding series of composite quotient group i.e.

$$G/H_1, H_1/H_2, ----, H_{n-1}/H_n$$

$$\text{and } G/K_1, \frac{K_1}{K_2}, ----, K_{m-1}/K_m$$

are abstractly identical i.e. one-one correspondence such that the corresponding quotient groups are isomorphic.

Proof :- We prove by the method of induction on the order of the group G. We first suppose that the theorem is true for all groups of order less than the order of G and then we shall prove that the theorem is true for G.

Clearly the theorem is true for all groups of order one

Now the cases arise:

Case I :- when $H_1 = K_1$. So $G/H_1 = G/K_1$

After removing G from ① and ②
we see that

$$H_1, H_2, \ldots, H_n = \{e\}$$
$$\text{and } K_1, K_2, \ldots, K_m = \{e\}$$

are two Composition series of H_1 and the corresponding series of Composition quotient groups are

$$H_1/H_2, H_2/H_3, \ldots H_{n-1}/H_n \text{ ----- } ③$$
$$\text{and } K_1/K_2, K_2/K_3, \ldots, K_{m-1}/K_m \text{ ----- } ④$$

But $o(H_1) < o(G)$. since H_2 is a proper normal subgroup of G.

As we have already supposed that the theorem is true for every group whose order is less than $o(G)$. So, the theorem is true for H_1. i.e. series ③ and ④ are isomorphic.

So the series (1') and (2') are isomorphic.

<u>Case II</u> when $H_1 \neq K_1$

$H_1 K_1$ is a normal subgroup of G Containing H_1 as well as K_1.

Moreover H_1 is a maximal normal subgroup of G

So: $H_1 K_1 = G$.

By third isomorphic theorem:

$$H_1K_1/H_1 \cong K_1/H_1 \cap K_1$$

and
$$H_1K_1/K_1 \cong H_1/H_1 \cap K_1$$

so: $G/H_1 \cong K_1/D$, where $D = H_1 \cap K_1$

and $G_1/K_1 \cong H_1/D$

Since H_1 is maximal in G so G/H_1 is simple.

Therefore K_1/D is also simple. This implies D is a maximal normal subgroup of K_1.

Similarly D is also a maximal normal subgroup of H_1.

Let $D, D_1, D_2, \ldots, D_i = \{e\}$ be a composition series for D. Then:

$$G, H_1, D, D_1, D_2, \ldots, D_i = \{e\} \quad \text{--- (5)}$$

and
$$G_1, K_1, D, D_1, D_2, \ldots, D_i = \{e\} \quad \text{--- (6)}$$

are two composition series of G.

Then the composition quotient groups are

$$G/H_1, H_1/D, D/D_1, D_1/D_2, \ldots, D_{i-1}/D_i \quad \text{--- (7)}$$

and
$$G/K_1, K_1/D, D/D_1, D_1/D_2, \ldots, D_{i-1}/D_i \quad \text{--- (8)}$$

The quotient groups in 7 and 8 are equal in numbers and the corresponding quotient groups are isomorphic.

$$A_1 : G/H_1 \cong K_1/D,\ H_1/D \cong G/K_1,\ D/D_1 \cong D/D_1$$

$$\ldots\ D_{i-1}/D_i \cong D_{i-1}/D_i$$

Now 1 and 5 are two composition series of G each having H_1 in second place. Therefore by Case I, the quotient groups defined by 1 and 5 may be put one-one correspond similarly the quotient groups defined by 2 and 6 may be put one-one correspondence. So that the corresponding quotient groups are isomorphic. Hence the quotient groups defined by 1 and 2 are equal in number and are isomorphic in same order, because the relation of isomorphism in the set of all groups is an equivalence relation

Proved.