Isomorphic vector space: A vector space U is said to be isomorphic to a vector space V over a field F, if there exists a mapping $T: U \to V$ such that

(i) T is linear transformation i.e
$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$
$$x, y \in U \ \& \ \alpha, \beta \in F$$

(ii) T is one-one i.e $T(x) = T(y) \Rightarrow x = y$ $\quad x, y \in U$

(iii) T is onto i.e for each $y \in V$, there exists some $x \in U$
$$T(x) = y, \ T \text{ is onto iff } V = T(x)$$

* Also, then the two vector spaces U and V are said to be isomorphic and symbollically write as $U(F) \cong V(F)$

This notation denotes that U is isomorphic to V.

Theorem - Any two finite-dimensional vector spaces over the same field are isomorphic.

Proof Let U and V be two finite-dimensional vector spaces over field F, such that $\dim U = \dim V = n$, Then prove that $U \cong V$ i.e U is isomorphic to V.

$T: U \to V$ as T is L.T, T is one-one and T is onto.

Let the sets of vectors $U = \{u_1, u_2, \dots u_n\}$ and $V = \{v_1, v_2, \dots v_n\}$ be the bases of U and V respectively. then $\dim U = \dim V$

any vector $u \in U$ can be expressed as

$u = \alpha_1 u_1 + \alpha_2 u_2 + \dots \alpha_n u_n$, $\alpha_i \in F$ & $u \in U$

$u' = \beta_1 u_1 + \beta_2 u_2 + \dots \beta_n u_n$ $\beta_i \in F$ & $u' \in U$

Let $T: U \to V$ be defined as

$$T(u) = v = \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n$$

Since in the expression for u as a linear combination of $u_1, u_2, \dots u_n$ with scalars $\alpha_1, \alpha_2 \dots \alpha_n$ are unique therefore the mapping T is well-defined i.e $T(u)$ is a unique element of vector space V.

(1) <u>T is one-one</u> We have $T(u) = T(u') \Rightarrow u = u'$

$T(\alpha_1 u_1 + \alpha_2 u_2 + \dots \alpha_n u_n) = T(\beta_1 u_1 + \beta_2 u_2 + \dots \beta_n u_n)$

$\alpha_i, \beta_i \in F$ & $u_i \in U$

for $i = 1, 2 \dots n$

$\alpha_1 T(u_1) + \alpha_2 T(u_2) + \dots \alpha_n T(u_n) = \beta_1 T(u_1) + \beta_2 T(u_2) + \dots + \beta_n T(u_n)$

$\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \dots \beta_n v_n$

$(\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + \dots + (\alpha_n - \beta_n) v_n = 0$

$\alpha_1 - \beta_1 = 0, \alpha_2 - \beta_2 = 0 \dots \alpha_n - \beta_n = 0$

$\because v_1, v_2, \dots v_n$ are L.I

$\Rightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots \alpha_n = \beta_n$

$\therefore \alpha_1 u_1 + \alpha_2 u_2 + \dots \alpha_n u_n = \beta_1 u_1 + \beta_2 u_2 + \dots \beta_n u_n$

$\therefore T$ is one-one.

(ii) T is onto V → If $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n$ is any element of V, then there exists an element $u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_n u_n \in U$ such that $T(u) = v$

$$T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_n u_n) = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n$$

$\therefore$ T is onto V.

(iii) T is a Linear Transformation. We have

$$T\{\alpha(\alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_n u_n) + \beta(\beta_1 u_1 + \beta_2 u_2 + \cdots \beta_n u_n)\}$$

$$T\{(\alpha\alpha_1 u_1 + \alpha\alpha_2 u_2 + \cdots \alpha\alpha_n u_n) + (\beta\beta_1 u_1 + \beta\beta_2 u_2 + \cdots \beta\beta_n u_n)\}$$

$$T\{(\alpha\alpha_1 + \beta\beta_1)u_1 + (\alpha\alpha_2 + \beta\beta_2)u_2 + \cdots + (\alpha\alpha_n + \beta\beta_n)u_n\}$$

$$(\alpha\alpha_1 + \beta\beta_1)T(u_1) + (\alpha\alpha_2 + \beta\beta_2)T(u_2) + \cdots (\alpha\alpha_n + \beta\beta_n)T(u_n)$$

$$(\alpha\alpha_1 + \beta\beta_1)v_1 + (\alpha\alpha_2 + \beta\beta_2)v_2 + \cdots (\alpha\alpha_n + \beta\beta_n)v_n$$

$$\alpha\alpha_1 v_1 + \beta\beta_1 v_1 + \alpha\alpha_2 v_2 + \beta\beta_2 v_2 + \cdots \alpha\alpha_n v_n + \beta\beta_n v_n$$

$$\alpha(\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n) + \beta(\beta_1 v_1 + \beta_2 v_2 + \cdots \beta_n v_n)$$

$$\alpha T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_n u_n) + \beta T(\beta_1 u_1 + \beta_2 u_2 + \cdots \beta_n u_n)$$

$$\alpha T(u) + \beta T(u')$$

$\therefore$ T is a linear transformation

Hence T is an isomorphism of U onto V

$$\therefore \underline{\underline{U \cong V}}$$