Invertible Linear Transformation (T^{-1}):

Let U and V be two vector spaces over a field F, and let $T: U \to V$ be a one to one and onto linear transformation. Then the inverse mapping of linear transformation, $T^{-1}: V \to U$ exists and is also a L.T. [illegible]

Let $v_1, v_2 \in V$ and $\alpha, \beta \in F$

Since $T: U \to V$ is onto, there exists $u_1, u_2 \in U$

$$T(u_1) = v_1 \quad \text{and} \quad T(u_2) = v_2$$

$$\Rightarrow u_1 = T^{-1}(v_1) \quad \text{and} \quad u_2 = T^{-1}(v_2)$$

We have $T^{-1}(\alpha v_1 + \beta v_2) = T^{-1}\{\alpha T(u_1) + \beta T(u_2)\}$

$= T^{-1}\{T(\alpha u_1 + \beta u_2)\}$ $\because T$ is Linear Transf.

$= T^{-1}T\{\alpha u_1 + \beta u_2\}$

$= I(\alpha u_1 + \beta u_2)$ $\because T^{-1}T = I$

$= \alpha u_1 + \beta u_2$

$\therefore T^{-1}(\alpha v_1 + \beta v_2) = \alpha T^{-1}(v_1) + \beta T^{-1}(v_2)$

Thus $T^{-1}: V \to U$ is Linear Transformation

* A one-one and onto linear transformation $T: U \to V$ is called an invertible (inverse) of T as $T^{-1}: V \to U$ is linear transformation

Non-Singular L.T. A Linear transformation $T: U \to V$ is called non-singular if the null space of T ($\ker T$) is zero or $\ker T = \{0\}$ or $N(T)$ consists of the zero vector alone i.e. if $u \in U$ and $T(u) = 0 \Rightarrow u = 0$ then $T(u) = 0 \Rightarrow u = 0$ and T is singular then $T(u) = 0 \Rightarrow u \neq 0$

Thus if T is non-singular

Also $u_1, u_2 \in U$ when T is non-singular $T(u_1) = 0$, $T(u_2) = 0$

$\Rightarrow T(u_1) = T(u_2) \Rightarrow T(u_1) - T(u_2) = 0 \Rightarrow T(u_1 - u_2) = 0 \Rightarrow u_1 = u_2$ [illegible]

Product of two Homomorphisms (L.T.)

Let $T_1, T_2 \in Hom(U,V)$. The product of T_1 and T_2

$T_1 T_2 : U \to V$ is defined as $(T_1 T_2)u = T_1(T_2(u)), \forall u \in U$

Remark (I) $T_1, T_2 \in Hom(U,V) \Rightarrow T_1 T_2 \in Hom(U,V)$

Let $x, y \in U$ & $\alpha, \beta \in F$ Then

$$T_1 T_2(\alpha x + \beta y) = T_1\{T_2(\alpha x + \beta y)\}$$

$$= T_1\{\alpha T_2(x) + \beta T_2(y)\} \quad \text{as } T_2 \text{ is Hom}$$

$$= \alpha T_1(T_2(x)) + \beta T_1(T_2(y)) \quad \text{as } T_1 \text{ is hom}$$

$$= \alpha (T_1 T_2)(x) + \beta (T_1 T_2)(y)$$

Hence $T_1 T_2 \in Hom(U,V)$

Remark if $T \in Hom(U,U)$, we write T^2 as TT

T^3 as $T^2 T$ etc.

Theorem if $T_1, T_2, T_3 \in Hom(U,U)$ then

(I) $IT = TI = T$, I is identity mapping of $Hom(U,U)$

(II) $T_1(T_2 + T_3) = T_1 T_2 + T_1 T_3$

(III) $(T_1 + T_2)T_3 = T_1 T_3 + T_2 T_3$

(IV) $\alpha(T_1 T_2) = (\alpha T_1)T_2 = T_1(\alpha T_2) \quad \alpha \in F$

(V) $T_1(T_2 T_3) = (T_1 T_2)T_3$