5[a]

Q. Prove that the necessary & sufficient conditions for a group G to be internal direct product of its two subgroups H & K are
(i) H & K are normal subgroups of G
(ii) $H \cap K = \{e\}$
(iii) $G = HK$

or

Prove that a group G is the internal direct product of its two subgroups H & K if and only if
(i) H & K are normal subgroups of G
(ii) $H \cap K = \{e\}$
(iii) $G = HK$

At first we suppose that the conditions (i), (ii) & (iii) hold. We shall prove that G is the internal direct product of H & K.

Let $h \in H$, $k \in K$. Then h & $k \in G$ also.
Since H is a normal subgroup of G, therefore
$khk^{-1} \in H \ \forall\ k \in G$ and $h \in H$
Also $h^{-1} \in H$. Therefore $khk^{-1}h^{-1} \in H$ —— (1)
Again K is a normal subgroup of G, therefore
$hk^{-1}h^{-1} \in K \ \forall\ h \in G$ & $k^{-1} \in K$
Also $k \in K$. Therefore $khk^{-1}h^{-1} \in K$ —— (2)
$\therefore$ from (1) & (2), we have $khk^{-1}h^{-1} \in H \cap K$
But from the condition (ii) of the theorem, $H \cap K = \{e\}$
$\therefore khk^{-1}h^{-1} = e$ i.e., $kh(hk)^{-1} = e$ i.e., $kh(hk)^{-1}(hk) = e(hk)$
i.e., $khe = hk$ i.e., $kh = hk$
Hence every element of H commutes with every element of K.

Let $x \in G$. Then by condition (iii) of the theorem there exist $h \in H$ & $k \in K$ such that $x = hk$.
If possible let $x = h_1k_1$ where $h_1 \in H$, $k_1 \in K$
Then $hk = h_1k_1 \Rightarrow hkk^{-1} = h_1k_1k^{-1} \Rightarrow he = h_1k_1k^{-1} \Rightarrow h = h_1k_1k^{-1}$
$\Rightarrow h_1^{-1}h = h_1^{-1}h_1k_1k^{-1} \Rightarrow h_1^{-1}h = ek_1k^{-1} \Rightarrow h_1^{-1}h = k_1k^{-1}$
But $h_1^{-1}h \in H$ & $k_1k^{-1} \in K$
$\therefore h_1^{-1}h = k_1k^{-1} \in H \cap K$
Since $H \cap K = \{e\}$, by condition (iii) of the theorem
$\therefore h^{-1}h = k_1k^{-1} \in \{e\} \Rightarrow h_1^{-1}h = k_1k^{-1} = e \Rightarrow h_1 = h$ & $k_1 = k$

$\therefore$ the expression $x = hk$ is unique.

i.e. any element of G can be expressed uniquely as the product of an element of H by an element of K.

Thus G is an internal direct product of its subgroups H & K.

Conversely let G is an internal direct product of its subgroups H & K. Then we shall show that the conditions (i), (ii) & (iii) hold.

Let $x \in G$. Then there exist $h \in H$ & $k \in K$ such that $x = hk$

Let $a \in H$. Now, $x a x^{-1} = (hk) a (hk)^{-1}$

$= hk a k^{-1} h^{-1} = h a k k^{-1} h^{-1} = h a h^{-1} \in H$

i.e. $x a x^{-1} \in H \; \forall \; x \in G$ & $a \in H$

$\therefore$ H is a normal subgroup of G.

Similarly we can prove that K is a normal subgroup of G.

Thus the condition (i) holds.

Let x be the common element of H & K. Then $x \in H$, $x \in K$

Since H & K are subgroups, therefore $x^{-1} \in H$, $x^{-1} \in K$

Since G is an internal direct product of H & K, therefore every element $g \in G$ can be uniquely expressed as $g = hk$, where $h \in H$, $k \in K$

Also $g = hk = h x x^{-1} k = (hx)(x^{-1}k)$, where $hx \in H$ & $x^{-1}k \in K$

Since this expression is unique, therefore $hk = (hx)(x^{-1}k)$ $\therefore h = hx, k = x^{-1}k$

$h = hx \Rightarrow he = hx \Rightarrow e = x$, by left cancellation law

i.e, $x = e$

Hence e is the only element common to both H & K i.e. $H \cap K = \{e\}$

Thus the condition (ii) holds

Now we have $HK \subseteq G$ — (3)

Also for any $x \in G$ there exist $h \in H$ & $k \in K$ such that

$x = hk \in HK$ i.e. $x \in HK$

Thus we get $x \in G \Rightarrow x \in HK$

$\therefore$ $G \subseteq HK$ — (4)

$\therefore$ from (3) & (4) we get $G = HK$

Thus the condition (iii) holds.

This completes the theorem.