Q. If a group G is an internal direct product of its subgroups H & K, then prove that

(i) H & K have only identity in common

(ii) G is isomorphic to the external direct product of H & K.

(i) Let x be the common element of H & K

$\therefore x \in H, x \in K$

Since H & K are subgroups, therefore $x^{-1} \in H, x^{-1} \in K$

Since G is an internal direct product of H & K, therefore every element $g \in G$ can be uniquely expressed as

$g = hk$ where $h \in H, k \in K$

Also $g = hk = hxx^{-1}k = (hx)(x^{-1}k)$, where $hx \in H$, $x^{-1}k \in K$

Since this expression is unique, therefore

$hk = (hx)(x^{-1}k)$

$\therefore h = hx, k = x^{-1}k$

Now $h = hx$ gives $he = hx$

$\Rightarrow e = x$, by left cancellation law

$\Rightarrow x = e$

Hence e is the common element to both H & K

i.e. $H \cap K = \{e\}$

(ii) Since G is the internal direct product of H & K, therefore every element of H commutes with every element of K.

i.e. $hk = kh \ \forall \ h \in H, k \in K$

and any $g \in G$ can be uniquely expressed as $g = hk$, where $h \in H, k \in K$

Let us define a mapping $f : G \to H \times K$ such that

$f(g) = f(hk) = (h, k)$

Let $g_1 = h_1k_1$ & $g_2 = h_2k_2$ be any two elements of G where $h_1, h_2 \in H$ & $k_1, k_2 \in K$

Then $f(g_1) = f(g_2) \Rightarrow f(h_1 k_1) = f(h_2 k_2)$
$\Rightarrow (h_1, k_1) = (h_2, k_2)$
$\Rightarrow h_1 = h_2,\ k_1 = k_2$
$\Rightarrow h_1 k_1 = h_2 k_2$
$\Rightarrow g_1 = g_2$

$\therefore f$ is one-one.

Also we see that every $(h, k) \in H \times K$ is the image of every $hk \in G$ under the mapping f

$\therefore f$ is onto

Again let $g_1 = h_1 k_1$ & $g_2 = h_2 k_2$ be any two elements of G.

Then $f(g_1) = f(h_1 k_1) = (h_1, k_1)$
& $f(g_2) = f(h_2 k_2) = (h_2, k_2)$

$\therefore f(g_1 g_2) = f(h_1 k_1 h_2 k_2)$
$= f(h_1 h_2 k_1 k_2)$, since $k_1 h_2 = h_2 k_1$
$= (h_1 h_2, k_1 k_2)$
$= (h_1, k_1)(h_2, k_2)$, by external direct product
$= f(g_1) f(g_2)$

$\therefore f$ is a homomorphism

Thus we see that f is an isomorphism of G onto $H \times K$

Hence G is isomorphic to the external direct product of H & K.

Q. If $G = H \times K$ is an external direct product of groups H & K, then show that G is abelian if & only if H & K are both abelian

Let $G = H \times K$ be abelian.

Let e_1 & e_2 be the identity elements of H & K respectively.

Let $(h_1, e_2), (h_2, e_2) \in G$ i.e. $H \times K$, where $h_1, h_2 \in H$ & $e_2 \in K$

Then G is abelian $\Rightarrow (h_1, e_2)(h_2, e_2) = (h_2, e_2)(h_1, e_2)$

$\Rightarrow (h_1 h_2, e_2 e_2) = (h_2 h_1, e_2 e_2)$, by external direct product

$\Rightarrow h_1 h_2 = h_2 h_1 \; \forall \; h_1, h_2 \in H$

$\Rightarrow H$ is abelian

Again let $(e_1, k_1), (e_1, k_2) \in G$ i.e. $H \times K$, where $e_1 \in H$ & $k_1, k_2 \in K$

Then G is abelian $\Rightarrow (e_1, k_1)(e_1, k_2) = (e_1, k_2)(e_1, k_1)$

$\Rightarrow (e_1 e_1, k_1 k_2) = (e_1 e_1, k_2 k_1)$

$\Rightarrow k_1 k_2 = k_2 k_1 \; \forall \; k_1, k_2 \in K$

$\Rightarrow K$ is abelian

Thus G is abelian $\Rightarrow H$ & K are both abelian

Conversely, let H & K be abelian & $G = H \times K$

Let $(h_1, k_1), (h_2, k_2) \in G$ i.e. $H \times K$

Then $h_1, h_2 \in H$ & $k_1, k_2 \in K$

$\therefore h_1 h_2 = h_2 h_1$ & $k_1 k_2 = k_2 k_1$, since H & K are abelian

$\therefore (h_1, k_1)(h_2, k_2) = (h_1 h_2, k_1 k_2)$, by external direct product

$= (h_2 h_1, k_2 k_1)$

$= (h_2, k_2)(h_1, k_1)$, by external direct product

Hence G is abelian.