Internal Direct Product:

Let G be a group and H, K be its subgroups. Then G is said to be the internal direct product of H & K if

(i) every element of H commutes with every element of K

(ii) every element of G is uniquely expressible as the product of an element of H by an element of K.

Q. If G is an internal direct product of its subgroups H_1 & H_2, then prove that

(i) H_1 & H_2 are normal subgroups of G

(ii) $G/H_1 \cong H_2$ and $G/H_2 \cong H_1$

Since G is an internal direct product of H_1 & H_2, then by definition

$$x_1 x_2 = x_2 x_1 \quad \forall\ x_1 \in H_1 \ \&\ x_2 \in H_2$$

and any $x \in G$ is uniquely expressible as

$$x = x_1 x_2 \text{ for some } x_1 \in H_1 \ \&\ x_2 \in H_2$$

Let us define a mapping $f : G \to H_1$ such that

$$f(x) = f(x_1 x_2) = x_1$$

Clearly f is onto.

Let $x = x_1 x_2$ & $y = y_1 y_2$ be any two elements of G

Then $f(x) = f(x_1 x_2) = x_1$

& $f(y) = f(y_1 y_2) = y_1$

Now $xy = x_1 x_2 \cdot y_1 y_2$
$= x_1 (x_2 y_1) y_2$
$= x_1 (y_1 x_2) y_2$
$= (x_1 y_1)(x_2 y_2)$, where $x_1 y_1 \in H_1$ & $x_2 y_2 \in H_2$

$\therefore f(xy) = f[(x_1 y_1)(x_2 y_2)] = x_1 y_1 = f(x) f(y)$

Hence f is a homomorphism of G onto H_1

By definition of Kernel, we have

$\text{Ker } f = \{x \in G : f(x) = e\}$, where e is the identity of G.

$= \{x_1 x_2 \in G : f(x_1 x_2) = e ; x_1 \in H_1, x_2 \in H_2\}$

$= \{x_1 x_2 \in G : x_1 = e ; x_1 \in H_1, x_2 \in H_2\}$

$= \{e x_2 \in G : x_2 \in H_2\}$

$= \{x_2 \in G : x_2 \in H_2\}$

$= H_2$

Thus we have proved that f is a homomorphism of G onto H_1 with $\text{Ker } f = H_2$

Hence H_2 is a normal Subgroup of G.

Also by fundamental theorem on Homomorphism of groups,

$$G/H_2 \cong H_1$$

Similarly we can show that a mapping $\phi : G \to H_2$ such that $\phi(x) = \phi(x_1 x_2) = x_2$ is a homomorphism onto with $\text{Ker } \phi = H_1$ which gives the result H_1 is a normal Subgroup of G and $G/H_1 \cong H_2$