Integrating Factors:-

We noted that the equation $y\,dx - x\,dy = 0$ is not exact diff. eqn, but that if we multiply by $1/y^2$ we have an exact equation then

$$\frac{dx}{y} - \frac{x\,dy}{y^2} = d\left(\frac{x}{y}\right) = 0$$

or by x/y^3

$$\Rightarrow \frac{x\,dx}{y^2} - \frac{x^2\,dy}{y^3} = d\left(\frac{1}{2}\frac{x^2}{y^2}\right) = 0$$

or by $1/xy$

$$\Rightarrow \frac{dx}{x} - \frac{dy}{y} = d\left(\log|x| - \log|y|\right) = 0$$

The multiplrers $\frac{1}{y^2}, \frac{x}{y^3}, \frac{1}{xy}$ are called integrating factors of the give equation.

* **Definition:-** A nonzero function $\mu(x,y)$ is called integrating factor of the equation

$$M\,dx + N\,dy = 0$$

if the equation

$$\mu(M\,dx + N\,dy) = 0$$

is exact.

Theorem 1:- The differential equation $M(x,y)dx + N(x,y)dy = 0$ —(1) has an infinite number of integrating factors.

Theorem 2:- If μ_1 and μ_2 are integrating factors of above diff. Eqn (1) whose ratio is not constant, the general solution of the equation (1) is

$$\mu_1 = c\mu_2$$

Exp. Solve the differential equation

$$\left(2y + \frac{1}{(x+y)^2}\right)dx + \left(3y + x + \frac{1}{(x+y)^2}\right)dy = 0$$

Solution- Given diff. Equation compare with

$$M(x,y)dx + N(x,y)dy = 0$$

$$M = 2y + \frac{1}{(x+y)^2} \quad \& \quad N = 3y + x + \frac{1}{(x+y)^2}$$

Assuming that μ is a function of $x+y$, we have

$$\frac{1}{N-M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x+y}$$

$$\frac{\partial M}{\partial y} = 2 - 2(x+y)^{-3} \cdot 1 = 2 - 2(x+y)^{-3}$$

$$\frac{\partial N}{\partial x} = 1 - 2(x+y)^{-3} \cdot 1 = 1 - 2(x+y)^{-3}$$

$$\frac{1}{N-M} = \frac{1}{3y + x + \cancel{\frac{1}{(x+y)^2}} - 2y - \cancel{\frac{1}{(x+y)^2}}} = \frac{1}{x+y}$$

then

$$\mu = \exp\left[\int \frac{1}{x+y}\, d(x+y)\right]$$

$$\mu = \exp[\log(x+y)] = x+y$$

Multiplying the original equation by $\mu = x+y$ we get

$$\left[2(x+y)y + \frac{1}{x+y}\right]dx + \left[(3y+x)(x+y) + \frac{1}{x+y}\right]dy = 0$$

This equation is exact

$$\frac{\partial}{\partial y}\left[2(x+y)y + \frac{1}{x+y}\right] = 2x+4y - \frac{1}{(x+y)^2}$$

$$\frac{\partial}{\partial x}\left[(3y+x)(x+y) + \frac{1}{x+y}\right] = 3y+2x+0+y - \frac{1}{(x+y)^2}$$

And the general integral is

$$\int_0^y \left[(3y+x)(x+y) + \frac{1}{x+y}\right]dy + \int_{x_0}^{x} \frac{1}{x}\,dx$$

$$= y^3 + x^2y + 2xy^2 + \log|x+y| - \log|x| + \log|x| - \log|x_0| = c$$

$$y^3 + x^2y + 2xy^2 + \log|x+y| = c_1$$

or $y^3 + x^2y + 2xy^2 + \log|x+y| = c_1$