Indeterminate form $[0^0, \infty^0, 1^\infty]$

Let $y = u^v$ — (1) where u & v are functions of x.

Taking logarithm of (1), we get

$$\log y = v \log u \quad \text{— (2)}$$

Case (i) if $u = 0, v = 0$ then $y = 0^0$

Case (ii) if $u = \infty, v = 0$ then $y = \infty^0$

Case (iii) if $u = 1, v = \infty$ then $y = 1^\infty$

Then from (2)

According to case (i) if $u = 0, v = 0$, then $\log y = 0 \times \infty$

case (ii) if $u = \infty, v = 0$ then $\log y = \infty \times 0$

Case (iii) if $u = 1, v = \infty$ then $\log y = \infty \times 0$

In these situations, $\log y$ is reduced to the indeterminate form $0 \times \infty$.

Exp. limit of $\left(1 + \frac{a}{x}\right)^x$ as $x \to \infty$

or $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$

Sol. let $y = \left(1 + \frac{a}{x}\right)^x$

$$\log y = x \log\left(1 + \frac{a}{x}\right)$$

$$= \lim_{x \to \infty} x \log\left(1 + \frac{a}{x}\right)$$

$$= \lim_{x \to \infty} \frac{\log\left(1 + \frac{a}{x}\right)}{1/x} \quad \left[\text{form } \frac{0}{0}\right]$$

$\lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0$

$$= \lim_{x \to \infty} \frac{\frac{1}{1 + \frac{a}{x}} \cdot \left(\cancel{-}\frac{a}{x^2}\right)}{\cancel{-}\frac{1}{x^2}} = \lim_{x \to \infty} \frac{a}{1 + a/x} = \frac{a}{1 + 0}$$

$$\log y = 9$$
$$y = e^9$$
$$\lim_{x\to\infty}\left(1+\frac{9}{x}\right)^x = e^9$$

Exp Evaluate $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$

Solution This is of form $[1^\infty]$ $\quad \because \lim_{x\to 0}\frac{\sin x}{x} = 1$, $\left(\frac{1}{x}\right)^2 = \infty$

let $y = \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

$$\log y = \frac{1}{x^2}\left(\log\frac{\sin x}{x}\right) \quad \text{—— (1)}$$

$$\because \sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \cdots$$
$$\frac{\sin x}{x} = 1 - \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} - \frac{x^6}{\lfloor 7} + \cdots$$
$$\log\frac{\sin x}{x} = \log\left(1 - \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} - \frac{x^6}{\lfloor 7} + \cdots\right)$$
$$= \log\left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right)\right]$$
$$= -\left(\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right) - \frac{1}{2}\left(\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right)^2 + \cdots$$
$$+ \frac{1}{3}\left(\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right)^3 - \cdots$$
$$= \frac{-x^2}{6} + x^4\left(\frac{1}{120} - \frac{1}{2}\cdot\frac{1}{36}\right) + \cdots$$
$$= \frac{-x^2}{6} - \frac{x^4}{180} + \cdots$$

From (1)

$$\log y = \frac{1}{x^2}\left[-\frac{x^2}{6} - \frac{x^4}{180} + \cdots\right]$$
$$= -\frac{1}{6} - \frac{x^2}{180} + \cdots$$
$$\lim_{x\to 0}\log y = \lim_{x\to 0}\left(-\frac{1}{6} - \frac{x^2}{180} + \cdots\right) = -\frac{1}{6} \Rightarrow y = e^{-1/6}$$