Maximal Ideal:- Let R be a ring and I an ideal in R. Then I is called a maximal ideal if I is properly contained in R and I is not properly contained in any other ideal of R.

Quotient ring

Let R be a ring & I an ideal in R. Then the set of all cosets of I in R denoted by R/I is defined as

$$R/I = \{I + a : a \in R\}$$

Let $I+a, I+b \in R/I$. If we define the operations of addition & multiplication on R/I as follows:

$$(I+a) + (I+b) = I + (a+b)$$
$$\& \ (I+a)(I+b) = I + ab,$$

then R/I is a ring w.r.t. the operations defined above. This ring is called quotient ring or factor ring.

Q. If I be an ideal of a commutative ring with unity, then prove that R/I is an integral domain if and only if I is prime ideal.

Let I be an ideal of a commutative ring R with unity.

We first suppose that I is a prime ideal. Then we shall prove that R/I is an integral domain.

We know that R/I is a ring w.r.t. the operations addition & multiplication defined below:

$$(I+a) + (I+b) = I + (a+b)$$
$$\& \ (I+a)(I+b) = I + ab \ \forall \ I+a, I+b \in R/I$$

Let $I+a,\ I+b \in R/I$. Then $a, b \in R$.

$$\therefore (I+a)(I+b) = I+ab$$
$$= I+ba$$
$$= (I+b)(I+a)$$

i.e. Commutative law for multiplication holds in R/I.

Also, $(I+a)(I+1) = I+a1$
$$= I+a$$
$$= (I+1)(I+a)$$

i.e. unity element exists in R/I

Now $(I+a)(I+b) = I \implies I+ab = I$

$\implies ab \in I$

$\implies a \in I$ or $b \in I$, since I is prime ideal

$\implies I+a = I$ or $I+b = I$

i.e. R/I has no zero divisors & hence R/I is an integral domain

Next we suppose that R/I is an integral domain.

We have to prove that I is prime ideal.

Since R/I is an integral domain, therefore R/I has no zero divisors

i.e. $(I+a)(I+b) = I \implies I+a = I$ or $I+b = I$

i.e. $I+ab = I \implies I+a = I$ or $I+b = I$

i.e. $ab \in I \implies a \in I$ or $b \in I$

This means that I is prime ideal.

This completes the proof.