Prime ideal:

Let R be a ring & I an ideal in R. Then I is called a prime ideal if whenever $ab \in I$, then either a belongs to I or b belongs to I.

Q. Show that a commutative ring with unity is a field if it has no proper ideals.

Let R be a commutative ring with unity such that R has no proper ideals.

Then R has only two ideals $\{0\}$ & R.

We have to prove that R is a field. For this we shall prove that every non-zero element of R has a multiplicative inverse in R.

Let $a \in R$ such that $a \neq 0$. Then we have to show that there exists an inverse of a in R.

We know that

$Ra = \{ra : r \in R\}$ is an ideal of R.

But by assumption,

$Ra = \{0\}$ or R

Now $1 \in R \Rightarrow 1a \in Ra$

$\Rightarrow a \in Ra$

$\Rightarrow Ra \neq \{0\}$

$\Rightarrow Ra = R$

Thus every element of R is a multiple of a by some element of R. In particular, $1 \in R$. So it can be realised as a multiple of a. Let there exists an element $b \in R$ such that $1 = ba$ $\therefore a^{-1} = b$ i.e. the inverse of a is b.

Hence each non-zero element of R has multiplicative inverse.

Therefore R is a field.

Q. Prove that a field R has no proper ideal

or

Prove that a field has only two ideals $\{0\}$ & R.

Let I be an ideal of a field R. To prove that R has no proper ideal we have to prove that R has only improper ideals i.e. We have to prove that $I = \{0\}$ or $I = R$.

If I be the set of only zero element, then $I = \{0\}$

Now we consider $I \neq \{0\}$

Then there exists $a \in I$ such that $a \neq 0$

$a \in I$ and $I \subseteq R \Rightarrow a \in R$

$\Rightarrow a^{-1} \in R$

Now $a \in I$ & $a^{-1} \in R \Rightarrow a^{-1}a \in I$

$\Rightarrow 1 \in I$

$\therefore$ any $x \in R$ and $1 \in I \Rightarrow 1x \in I$

$\Rightarrow x \in I$

Thus $x \in R \Rightarrow x \in I$ which shows that $R \subseteq I$ —— (1)

Since I is an ideal of R

$\therefore I \subseteq R$ —— (2)

from (1) & (2) we get

$I = R$

This proves that a field R has no proper ideals