Ideals

A non-empty subset I of a ring R is said to be an ideal of R if (i) $a+(-b) \in R \ \forall \ a, b \in I$
and (ii) ar & ra both $\in I$ for every $a \in I$ & $r \in R$

Q. Prove that the intersection of two ideals of a ring R is an ideal of R.

Let I_1 & I_2 be two ideals of a ring R

Let $a, b \in I_1$. Then by definition $a+(-b) \in I_1$

Let $a, b \in I_2$ also. Then $a+(-b) \in I_2$

$\therefore a+(-b) \in I_1 \cap I_2$ —— (1)

Let $r \in R$. Then $ar, ra \in I_1$ & I_2 both (by definition)

$\therefore ar, ra \in I_1 \cap I_2$ —— (2)

from (1) & (2) we conclude that $I_1 \cap I_2$ is an ideal of R.

Q. Prove that the intersection of any collection of ideals in a ring R is also an ideal of R.

Let R be a ring & S_α be a collection of ideals of R. Let $S = \cap S_\alpha$

Since every ideal is a subring, therefore each S_α is a subring. Therefore $\cap S_\alpha$ i.e. S is also a subring

Let $a \in S_\alpha$ for each α and $r \in R$

Since S_α is ideal of R, therefore $ar \in S_\alpha$ for each α

$\Rightarrow ar \in \cap S_\alpha$

$\Rightarrow ar \in S$

Similarly we can show that $ra \in S$

i.e. $a \in S, r \in R \Rightarrow ar \in S$ and $ra \in S$

Again, $a, b \in S_\alpha$ for each $\alpha \Rightarrow a, b \in S$

& $b \in S_\alpha \Rightarrow -b \in S_\alpha$

$\therefore a + (-b) \in S_\alpha$ for each α

$\Rightarrow a + (-b) \in S$

Thus, $a, b \in S \Rightarrow a + (-b) \in S$

Hence S is an ideal of R.

Ideal generated by an element of a ring:

Let R be a ring and $a \in R$. Then the ideal generated by a denoted by $[a]$ is defined as

$$[a] = \{sa : s \in R\} \subseteq R$$

Let us consider the ring of integers

$$Z = \{ \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots \}$$

Then $[4] = \{ \cdots, -12, -8, -4, 0, 4, 8, 12, \cdots \} \subseteq Z$

Principal ideal:

If an ideal I is generated by a single element $a \in R$ then I is called a principal ideal.

Symbolically, the principal ideal generated by a is denoted by $[a]$

Proper & Improper ideals:

In any ring R the subring $\{0\}$ & the whole ring R are also ideals. These are called improper ideals. Any other ideal is called proper ideal.