GROUP THEORY

HOMOMORPHISM OF A GROUP :-

Let $G = \{a, b, c, \ldots\}$ be a group with respect to the binary operation o and $G' = \{a', b', c', \ldots\}$ be another group with respect to $*$.

A mapping f of G onto G' is said to be homomorphism if $\forall\ a, b \in G$

$$f(a \circ b) = f(a) * f(b)$$

Notes :- (i) The word 'homomorphism' is derived from the two Greek words 'homos' meaning 'link' and morphic meaning 'form'.

(ii) A homomorphism f from G to G' carries the product $x \circ y$ in G to the product $f(x) * f(y)$ in G_2.

(iii) $f(G)$, if exists, is called homomorphic image of G.

(iv) The relation of homomorphism is expressed by writing

$$G = G'$$

If G is homomorphic to G', then there may exist more than one homomorphism of G and G'.

(1) Epimorphism :- A homomorphism f which is also onto is called an epimorphism

② **Monomorphism** :- A homomorphism f which is also one-one is called a monomorphism.

③ **Endomorphism** :- A homomorphism of a group into itself is called an endomorphism.

④ **Natural homomorphism** :- Let H be a normal subgroup of group G then the map $f: G \to G/H$ such that $f(x) = Hx$ is a homomorphism. This homomorphism is called natural or Canonical homomorphism.

Examples :-

(i) If $f(x) = e \ \forall \ x \in G$, this is homomorphism.

(ii) If G be a group of all real numbers under addition i.e. $a * b$ for $a, b \in G$ is the real number $a+b$ and G' be the multiplicative group of non-zero real numbers and

$$f: G \to G' \text{ defined by } f(a) = 2^a$$

is a homomorphism

$$[\because f(a+b) = 2^{a+b} = 2^a \cdot 2^b = f(a) \cdot f(b)$$

(iii) If G be the group of integers under addition and $G' = G$ and let $f(x) = 2x$ $\forall \ x \in G$, then

$$f(x+y) = 2(x+y) = 2x + 2y$$
$$= f(x) + f(y)$$

$\Rightarrow f$ is a homomorphism.

(iv) If G be the group of integers under addition and G'_n be the group of integers under addition modulo n and let f be defined as $f(x)$ = remainder of x on division by n, then it is a homomorphism.

(v) Let G be a group and let e be the identity element of G. Then the mapping $f : G \to G$ defined by $f(a) = e \ \forall \ a \in G$ is an endomorphism of G.

Kernel of a homomorphism :- Let f be a homomorphism of a group G into a group G', then the set K of all those elements of G, which are mapped by f onto the identity e' of G' is called the Kernel of a homomorphism.

Isomorphism :- A mapping f from a group (G, o) into $(G', *)$ is called an isomorphism if f is one-one, onto and

$$f(a * b) = f(a) * f(b) \ \forall \ a, b \in G$$

Notes :-

(i) The word isomorphism is derived from the Greek word 'isos' meaning 'equal'.

(ii) An isomorphism of a group G onto itself is called an automorphism of G.