Abstract Algebra

Homomorphism of Rings

Definition :- Let R and R' be two rings with two binary operations addition and multiplication, then a mapping ϕ from the ring R into the ring R'. is said to be a homomorphism if

(i) $\phi(a+b) = \phi(a) + \phi(b)$

(ii) $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in R$

Notes :-

(i) If ϕ is a homomorphism of R into R', then

(a) $\phi(0) = 0'$. $0 \in R$, $0' \in R'$

(b) $\phi(-a) = -\phi(a)$ $\forall\, a \in R$

(ii) Every homomorphic image of a Commutative ring is a Commutative ring.

(iii) If ϕ is a homomorphism of a ring R into a ring R', then $\phi(R)$ is a subring of R'.

Kernal of a Ring homomorphism

Definition Let R and R' be two rings and ϕ be a homomorphism from a ring R into a ring R', then the set of all those elements of R' which are mapped onto the zero element of R' is said to be kernel of ϕ and is usually denoted by $\text{Ker}\,\phi$.

i.e. if $0'$ is the zero element of R', then

$$\text{Ker}(\phi) = \{a \in R : \phi(a) = 0'\}$$

Notes:-

(i) If ϕ is a homomorphism of ring R into a ring R' with kernel S, then S is an ideal of R.

Isomorphisms and quotient rings

Definition A homomorphism ϕ from a ring R into a ring R' is said to be isomorphism if

(i) ϕ is one to one

(ii) ϕ is onto

Definition Let R be a ring and S be any ideal of R, then the algebraic structure $[R/S, +, \cdot]$ where $R/S = \{a+S : a \in R$ and the operations '$+$' and '$\cdot$' on R/S defined by

$$(a+S)+(b+S) = (a+b)+S$$

$$\text{and } (a+S).(b+S) = a.b+S$$

for all $a, b \in R$ forms a ring. This ring is called the Quotient ring of R w.r.t. the ideal S. The quotient ring is also known as residue class ring.