Abstract Algebra

Homomorphisms and Isomorphisms of Modules

Let M and N be two R-modules then a mapping T from M into N is called a homomorphism if

(i) $T(m_1 + m_2) = T(m_1) + T(m_2)$

(ii) $T(rm_1) = rT(m_1)$

for all $m_1, m_2 \in M$ and $r \in R$.

If the homomorphism T is one-one and onto then T is called an isomorphism

Kernel of Homomorphism

Let T be a homomorphism from a module M into a module N. Then the set of all those elements of M such that $T(m) = 0$ where 0 is the identity element of N is called Kernel of T. It is denoted by $K(T)$.

$$\therefore K(T) = \{m \in M : T(m) = 0\}$$

Range of Homomorphism

If T is a homomorphism from module M into module N. Then the set of all those elements of N which are the T image of the elements of M, are called Range of T. It is usually denoted by $R(T)$. So that

$$R(T) = \{T(m) \in N : m \in M\}$$

(i) The homomorphism $f: M \to N$ is also known as module homomorphism or R-module homomorphism.

(ii) R-module homomorphism is an R-module monomorphism if it is injective (one-one).

(iii) R-module homomorphism is an R-module epimorphism if it is surjective (onto),

(iv) R-module homomorphism is an R-module isomorphism if it is bijective (one-one onto).

(v) If $M = N$, then R-module homomorphism is an R-module endomorphism.

(vi) If M is isomorphic to N, then we can write $M \cong N$.

Note :-

(i) Kernel of homomorphism of module is a submodule.

(ii) The range of homomorphism of module is also submodule.

(iii) If T is a module homomorphism, then T is an isomorphism if and only if $K(T) = 0$

First theorem of isomorphism

If T is a homomorphism of M onto N with $K(T) = A$, prove that N is isomorphic to M/A (quotient module)

Proof :- Let ϕ be an mapping from M/A to N given by

$$\phi(x+A) = T(x)$$

Now we show that ϕ is isomorphic.

Let $x+A$, $y+A \in M/A$ for $x, y \in M$ and suppose that

$$\phi(x+A) = \phi(y+A) \Rightarrow T(x) = T(y)$$
$$\Rightarrow T(x) - T(y) = 0$$
$$\Rightarrow T(x-y) = 0 \Rightarrow x-y \in K(T)$$
$$\Rightarrow x-y \in A \quad [\because K(T) = A]$$
$$\Rightarrow x+A = y+A \text{ (By the definition of Coset)}$$
$$\Rightarrow \phi \text{ is one-to-one}$$

Also for $x \in M$, there exists $T(x) \in N$, because T is onto homomorphism and corresponding to $x \in M$, $x+A \in M/A$ such that $\phi(x+A) = T(x)$. This shows that ϕ is onto.

For $x+A$, $y+A \in M/A$

$$\phi\big((x+A)+(y+A)\big) = \phi(x+y+A) = T(x+y)$$
(By the definition of ϕ)
$$= T(x) + T(y) \quad \because T \text{ is (homomor}$$
$$= \phi(x+A) + \phi(y+A)$$

Also for $x+A \in M/A$ and $r \in R$, then

$$\phi(r(x+A)) = \phi(rx+A)$$
$$= T(rx) = r\,T(x)$$
$$(\because T \text{ is homomorphism})$$
$$= r\phi(x+A)$$

Hence ϕ is an isomorphism.