Group theory
Abstract Algebra

Cosets:- Let G be a group and H is any subgroup of G.

Let $a \in G$. Then the set $Ha = \{ha : h \in H\}$ is called right coset of H in G generated by a and the set

$a \cdot H = \{a \cdot h : h \in H\}$ is called left coset of H in G generated by a.

Clearly Ha and aH are both subsets of G.

If e is the identity element of G.
Then $He = H = eH$.

So, H is itself a right as well as left coset.

Normal subgroup:- A subgroup H of a group G is said to be normal subgroup of G if for every $x \in G$ and for every $h \in H$, we have $x \cdot h \cdot x^{-1} \in H$

So, a subgroup H of a group G is said to be a normal subgroup of G if $xHx^{-1} \subseteq H$; $\forall\ x \in G$.

Clearly H is a normal subgroup of G iff $xHx^{-1} = H\ \forall\ x \in G$.

Quotient Group :- Let H be any normal subgroup of a group G. If $a \in G$, then Ha is a right coset of H in G. As H is normal in G, so left coset aH will be equal to right coset Ha. So there is no difference between left coset and right cosets. So we say only coset of H in G.

Let G/H be the collection of all cosets of H in G.

i.e. $G/H = \{Ha : a \in G\}$

The set G/H is a group w.r. to multiplication of complexes as the composition. This group is called quotient group.

Definition :- If G is a group and H is a normal subgroup of G, then the set G/H of all cosets of H in G is a group w.r. to multiplication of cosets. It is called quotient group or factor group of G by H.

The identity element of quotient group G/H is H.

Maximal Normal Subgroup :-

A normal subgroup H of a group G is said to be maximal if there exists no proper normal subgroup K of G which properly Contains H.

i.e. a normal subgroup H of a group G is maximal if and only if there exists no normal subgroup K of G such that $H \subset K \subset G$.

Simple group :- A group G is said to be simple if it possesses no proper normal subgroups.

Theorem :- Let G be a group and H a normal subgroup of G. If K is a normal subgroup of G Containing H i.e. $H \subset K$ then the quotient group K/H is a normal subgroup of the quotient group G/H.

Conversely, if K/H is a normal subgroup of G/H, then K is a normal subgroup of G Containg H.

Proof :- It is given that H is a normal subgroup of ~~G/H~~ G ~~then~~ and K is a normal subgroup of G such that $H \subset K$ so H is also a normal subgroup of K and therefore K/H is a quotient group.

Now, if Ha be any element of K/H then $a \in K$.

As $K \subseteq G$ so $a \in K \Rightarrow a \in G$

Therefore Ha is also an element of G/H

So, $K/H \subseteq G/H$

Therefore K/H is a subgroup of G/H.

Let $Hg \in G/H$ and $Hk \in K/H$.

Then $g \in G$ and $k \in K$

Now $[Hg][Hk][Hg]^{-1} = (Hg)(Hk)(Hg^{-1})$, since $(Hg)^{-1} = Hg^{-1}$

$= Hgkg^{-1}$, since H is normal $\Rightarrow (Ha)(Hb) = Hab$

since K is a normal subgroup of G.

So, $g.k.g^{-1} \in K$

Hence $Hgkg^{-1} \in K/H$

Thus K/H is a normal subgroup of G/H.

Conversely, suppose K/H is a normal subgroup of G/H.

Let x be any element of G and K be any element of K

Then : $Hx \in G/H$ and $Hk \in G/K$

Since K/H is normal subgroup of G/H.

So : $(Hx)(H_k)(Hx^{-1}) \in K/H$

$\Rightarrow HxKx^{-1} \in K/H$, since H is normal in G.

$\Rightarrow xKx^{-1} \in K$

$\therefore K$ is normal in G.

Also K/H is a quotient group, so H is a normal subgroup of K. Hence K is normal subgroup of G and $H \subseteq K$. Proved