Graeffe's Root-Squaring Method:-

This is another method for the numerical solution of polynomial equations. The method is outlined here by considering a cubic equation.

Let the cubic equation be.

$$A_0x^3+A_1x^2+A_2x+A_3=0 \quad \text{---}(1)$$

Whose roots x_1, x_2 and x_3 are such that

$|x_1| >> |x_2| >> |x_3|$

The symbol $>>$ means "much greater than".

In otherwords, the ratios $\frac{x_2}{x_1}, \frac{x_3}{x_2}$ are very small quantities compared to unity (1) and can be neglected.

We now consider the relations between the roots and coefficients of equation (1).

$$\left.\begin{aligned} x_1+x_2+x_3 &= -\frac{A_1}{A_0} \\ x_1x_2+x_2x_3+x_3x_1 &= \frac{A_2}{A_0} \\ \text{and} \quad x_1x_2x_3 &= -\frac{A_3}{A_0} \end{aligned}\right\} \text{---}(2)$$

Since $\frac{x_2}{x_1}, \frac{x_3}{x_2}$ are negligible, the above relations give $\rightarrow$

$$\left.\begin{aligned} x_1 &= -\frac{A_1}{A_0} \\ x_2 &= -\frac{A_2}{A_1} \\ \& \; x_3 &= -\frac{A_3}{A_2} \end{aligned}\right\} \text{———— (3)}$$

Thus, the magnitudes of the roots will be known when once the roots are widely separated. Now, we shall show the way, the roots are separated by considering the cubic equation

$$p(x) = (x-1)(x-2)(x-3) \quad \text{———— (4)}$$

then

$$p(-x) = (-x-1)(-x-2)(-x-3) \quad \text{—— (5)}$$

$$p(-x) = (-1)^3(x+1)(x+2)(x+3) \quad \text{—— (5)}$$

therefore,

$$p(x)\,p(-x) = (-1)^3(x^2-1)(x^2-4)(x^2-9) \quad \text{— (6)}$$

letting

$$q(z) = (z-1)(z-4)(z-9) \quad \text{—— (7)}$$

where $z = x^2$, we find that the roots of eq^n. (7) are the squares of the roots of eq^n (4). By transforming equation (7) in the same way as above, we get another equation whose roots are the squares of the roots of eq^n (7).

This is the principle underlying this method and due to this reason, this method is called root-squaring method.

Now, let the given cubic be

$$f(x) = a_0x_3 + a_1x_2 + a_2x + a_3 = 0 \quad \text{— (8)}$$

with roots α_1, α_2 and α_3 such that

$$|\alpha_1| > |\alpha_2| > |\alpha_3|$$

Suppose that Eqn. (8) is transformed 'm' times by the root-squaring process described above, and that the transformed equation is

$$\phi(u) = a_0^m u^3 + a_1^m u^2 + a_2^m u + a_3^m = 0 \quad \text{— (9)}$$

If u_i are the roots of equation (9), then we have

$$u_i = \alpha_i^m \qquad i = 1, 2, 3 \quad \text{——— (10)}$$

But u_i are given by the coefficients in eqn (9). Hence, we have the following formulae for the roots of eqn (8).

$$\alpha_i = \left(\frac{a_i}{a_{i-1}}\right)^{1/m} \qquad i = 1, 2, 3 \quad \text{—— (11)}$$

These results can easily be generalized to a nth degree polynomial. This method gives approximations to the magnitudes of the roots. To find the sign of any roots, we substitute the root in the original polynomial and find the result.