* Generalized Newton's method:

If ξ is a root of $f(x)=0$ with multiplicity p, then the iteration formula corresponding to Newton-Raphson formula is taken as

$$x_{n+1} = x_n - p\frac{f(x_n)}{f'(x_n)} \quad \text{———} \quad (8)$$

which means that $(\frac{1}{p}) f'(x_n)$ is the slope of the straight line passing through (x_n, y_n) and intersecting the x-axis at the point $(x_{n+1}, 0)$.

Equation (8) is called the generalized Newton's formula and reduces to Eqn (1) for $p=1$.

Since ξ is root of $f(x)=0$ with multiplicity p, it follows that ξ is also a root of $f'(x)=0$ with multiplicity $(p-1)$, of $f''(x)=0$ with multiplicity $(p-2)$, and so on. Hence the expressions

$$x_0 - p\frac{f(x_0)}{f'(x_0)}, \quad x_0-(p-1)\frac{f'(x_0)}{f''(x_0)},$$

$$x_0-(p-2)\frac{f''(x_0)}{f'''(x_0)}, \dots$$

must have the same value if there is a root with multiplicity p, provided that the initial approximation x_0 is chosen sufficiently close to the root.

Exp. Find a double root of the equation

$$f(x) = x^3 - x^2 - x + 1 = 0.$$

Solution - Choosing $x_0 = 0.8$ and $p=2$

we have $f(x) = x^3 - x^2 - x + 1$

$$f'(x) = 3x^2 - 2x - 1$$

$$f''(x) = 6x - 2$$

With $x_0 = 0.8$, we obtain

$$x_0 - 2\frac{f(x_0)}{f'(x_0)} = 0.8 - 2\frac{(0.8)^3-(0.8)^2-(0.8)+1}{3(0.8)^2-2(0.8)-1}$$

$$x_0 - 2\frac{f(x_0)}{f'(x_0)} = 0.8 - 2\frac{0.072}{(-0.68)} = 1.012$$

and

$$x_0 - (p-1)\frac{f'(x_0)}{f''(x_0)} = 0.8 - (2-1)\frac{(-0.68)}{2.8}$$

$$x_0 - \frac{f'(x_0)}{f''(x_0)} = 0.8 - \frac{(-0.68)}{2.8} = 1.043$$

The closeness of these values indicates that there is a double root near to unity. For the next approximation, we choose $x_1 = 1.01$ and obtain

$$x_1 - 2\frac{f(x_1)}{f'(x_1)} = 1.01 - 2\frac{(1.01)^3 - (1.01)^2 - (1.01)+1}{3(1.01)^2 - 2(1.01) - 1}$$

$$x_1 - 2\frac{f(x_1)}{f(x_1)} = 1.01 - 0.0099 = 1.0001$$

and $$x_1 - \frac{f'(x_1)}{f''(x_1)} = 1.01 - \frac{3(1.01)^2 - 2(1.01) - 1}{6(1.01) - 2}$$

$$= 1.01 - 0.0099 = 1.0001$$

Thus, there is a double root at $x = 1.0001$ which is sufficiently close to the actual root unity.

On the other hand, if we apply Newton-Raphson method with $x_0 = 0.8$

$$x_1 = x_0 - \frac{f(x_0)}{f(x_0)} = 0.8 - \frac{(0.8)^3 - (0.8)^2 - (0.8)+1}{3(0.8)^2 - 2(0.8)^2 - 1}$$

$$= 0.8 + 0.106 = 0.906 \simeq 0.91$$

$$x_2 = x_1 - \frac{f(x_1)}{f(x_1)} = 0.91 - \frac{(0.91)^3 - (0.91)^2 - (0.91)+1}{3(0.91)^2 - 2(0.91)^2 - 1}$$

$$= 0.91 + 0.046 = 0.956 \simeq 0.96$$

It is clear that generalized Newton's method converges more rapidly than the Newton-Raphson procedure.