To find the solution of Linear differential equation, operator 'D' plays every important role. 'D' is defined as-

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \ldots \quad D^n = \frac{d^n}{dx^n}$$

$$\therefore \quad Dy = \frac{dy}{dx}; \quad D^2y = \frac{d^2y}{dx^2}; \ldots\ldots \quad D^ny = \frac{d^ny}{dx^n}$$

The n^{th} order linear differential Equation is defined as:-

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots P_ny = f(x) \quad \text{---(i)}$$

where $P_1, P_2, \ldots P_n$ are function of x or constants.

In Eqn (1) if $f(x) = 0$ then Linear equation is called homogeneous, otherwise it said to be non-homogeneous. differential equation.

Solution of linear differential equation (1) canbe separated into two parts.

(a) $P_1, P_2; P_n$ are constants

(b) $P_1, P_2, \cdots P_n$ are functions of x.

Since the general solution of n^{th} order differential equation contains n arbitrary constants, it follows from the above, that if $y_1, y_2, \cdots y_n$ are n solution of differential equation

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} \ldots\ldots P_ny = 0 \quad \text{---(2)}$$

then $c_1y_1 + c_2y_2 + \cdots c_ny_n \; (= u)$ is a solution of this Eqn.

Rule to find the Complementary function:

Consider the linear diff. Eqn.

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots k_n y = 0 \quad (1)$$

Where $k_1, k_2, k_3 \cdots k_n$ are arbitrary constants.

Then equation (1) can write in symbolic form -

$$(D^n + k_1 D^{n-1} + \cdots k_n)\, y = 0$$

Its symbolic co-efficient equated to zero

i.e

$$D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots k_n = 0$$

is called the auxiliary Equation.

Since it is nth order polynomial equation in terms of D, it has n roots like as m_1, m_2 --- m_n.

Case-I:- If all roots be real and different of auxiliary equation, then general solution of Eqn (1) is given by

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots c_n e^{m_n x}.$$

Case-II:- If two roots are equal (i.e $m_1 = m_2$) the solution

$$y = (c_1 + c_2 x) \cdot e^{m_1 x} + c_3 e^{m_3 x} + \cdots c_n e^{m_n x}.$$

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If, auxiliary equation has three equal roots i.e. $(m_1 = m_2 = m_3)$ the general solution is

$$y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \cdots \cdots c_n e^{m_n x}$$

Case–III :- If one pair of roots be imaginary i.e $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then general solution is given by

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \cdots c_n e^{m_n x}.$$

Case–IV :- If two pairs of imaginary roots be equal i.e $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$ then the general solution is as

$$y = e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right] + c_5 e^{m_5 x} + \cdots\cdots c_n e^{m_n x}.$$

Exp. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

Sol. The given Eqn writes to

$$(D^2 + D - 2)y = 0$$

Then its auxiliary Eqn is

$$D^2 + D - 2 = 0 \quad \text{i.e } (D+2)(D-1) = 0$$

$$\Rightarrow D = -2 \text{ and } D = 1$$

Thus general solution is $\Rightarrow$ $y = c_1 e^{-2x} + c_2 e^{x}$

Exp. Solve $\left(\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4\right) = 0$

Solution – The given Equation reduces to

$$(D^3 + D^2 + 4D + 4)\, y = 0$$

Hence the auxilrary equation is

$$D^3 + D^2 + 4D + 4 = 0$$

$$D^2(D+1) + 4(D+1) = 0$$

$$(D+1)(D^2+4) = 0 \Rightarrow D+1 = 0 \text{ or } D^2+4=0$$

$$\therefore D = -1, \pm 2i \qquad \alpha \pm i\beta = 0 \pm 2i$$

Thus, the general solution is

$$y = c_1 e^{-x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x)$$

$$= c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x$$

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