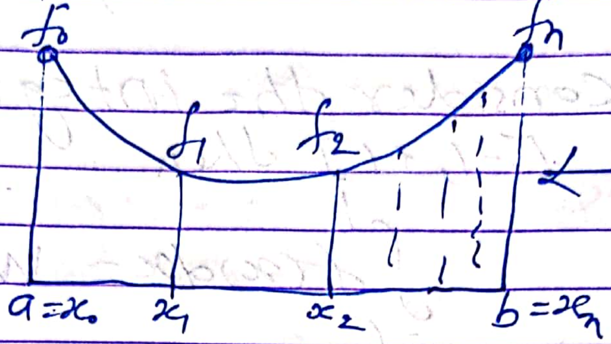
Gauss Quadrature:-

In general for Newton - Cotes (equispaced interpolation point/integration points)

$$\int_a^b f(x)\,dx = h[W_0 f_0 + W_1 f_1 + \cdots W_n f_n] + E$$



Closed formula

→ Note that for Newton-Cotes formulae only the weighting coefficients W_i were unknown and the x_i are fixed.

→ However, the number of and placement of the integration points influences the accuracy of the Newton-Cotes formulae.

* n even → n^{th} degree interpolation function exactly integrates an $(n+1)^{th}$ degree polynomial.
* n odd → n^{th} degree interpolation function exactly integrates an n^{th} degree polynomial.

→ Assume that for Gauss Quadrature the form of the integration rule is

$$\int^b$$

* Derive 1 point Gauss Quadrature
→ 2 unknowns W_0, x_0 which will exactly integrate any linear function.
→ Let the general polynomial be

$$f(x) = Ax + B \quad \text{--- (1)}$$

where A, B are equal value coefficients

→ Also, consider the integration interval to be $[-1, +1]$ then

$$\int_{-1}^{1} f(x)\,dx = W_0 f(x_0)$$

→ Substituting in the form of f(x) from Eq(1) then

$$\int_{-1}^{1} (Ax + B)\,dx = W_0 (Ax_0 + B)$$

$$\left[A\frac{x^2}{2} + Bx\right]_{-1}^{1} = W_0 (Ax_0 + B)$$

$$\left[A\left(\frac{1}{2}\right) + B(1) - A\left(\frac{1}{2}\right) - B(-1)\right] = A(W_0 x_0) + BW_0$$

$$A\left(\frac{1}{2} - \frac{1}{2}\right) + B(1+1) = A(W_0 x_0) + B(W_0)$$

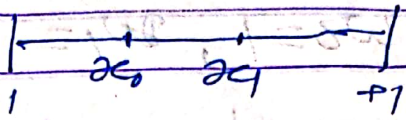
$$A(0) + B(2) = A W_0 x_0 + B W_0$$

In order for this to be true for any 1st degree polynomial (i.e. any A and B).

$$\begin{cases} 0 = x_0 W_0 \\ 2 = W_0 \end{cases}$$

Therefore $x_0 = 0$ and $W_0 = 2$ for point $(n=1)$ Gauss quadrature. We can integrate with only 1 point for a linear function.

* Derive 2 Points Gauss Quadrature:- formula



1) The general form of the integration formula is—

$$\int_{-1}^{1} f(x)\,dx = I = w_0 f_0 + w_1 f_1 \longrightarrow (1)$$

II) w_0, w_1, x_0, x_1 are all unknowns

III) 4 unknowns → we can fit a 3rd degree polynomial exactly.

$$f(x) = Ax^3 + Bx^2 + Cx + D$$

iv) on substituting $f(x)$ in Eqn (1)

$$I = \int_{-1}^{1} [Ax^3 + Bx^2 + Cx + D]\,dx = w_0 f(x_0) + w_1 f(x_1)$$

$$\left[A\frac{x^4}{4} + B\cdot\frac{x^3}{3} + C\frac{x^2}{2} + Dx\right]_{-1}^{1} = w_0[Ax_0^3 + Bx_0^2 + Cx_0 + D] + w_1[Ax_1^3 + Bx_1^2 + Cx_1 + D]$$

$$\frac{A}{4}(1-1) + \frac{B}{3}(1+1) + \frac{C}{2}(1-1) + D(1+1)$$

$$= A(w_0x_0^3 + w_1x_1^3) + B(w_0x_0^2 + w_1x_1^2) + C(w_0x_0 + w_1x_1) + D(w_0 + w_1)$$

$$w_0x_0^3 + w_1x_1^3 = 0$$

$$w_0x_0^2 + w_1x_1^2 = \frac{2}{3}$$

$$w_0x_0 + w_1x_1 = 0$$

$$w_0 + w_1 = 2$$

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Hence 4 non linear equation & 4 unknowns then

$$w_0 = 1, \quad w_1 = 1$$

$$x_0 = -\frac{1}{\sqrt{3}} \text{ and } x_1 = \frac{1}{\sqrt{3}}$$

Thus all polynomials of degree 3 or less will be exactly integrated with Gauss-Legendre point formula.

$$I = \int_{-1}^{1} f(x)\,dx = \sum_{i=0}^{N} w_i f_i + E$$

N	N+1	x_i, $i=0,N$	w_i	Exact for Polynomial degree
0	1	0	2	1 - (2n+1)
1	2	$-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$	1, 1	3
2	3	-0.774597, 0	0.5555, 0.8889	5
		+0.774597	0.5555	
3	4	-0.8611363?	0.34785485	7
		-0.33998104	0.65214575	
		+0.86114597	0.65214575	
		+0.33998104	0.34785485	
(N)	(N+1)	- -	- -	(2N+1)