*Newton - Cotes Integration formulae.

Let the interpolation points x_i be equally spaced i.e. $x_i = x_0 + ih$. $i = 0, 1, 2, \cdots n$ and let the end points of the interval of integration be placed such that

$$x_0 = a, \quad x_n = b \quad \& \quad h = \frac{b-a}{n}$$

Then the definite Integral

$$I = \int_a^b y\,dx$$ is evaluated by an integration

$$I_n = \sum_{i=0}^{n} c_i y_i \qquad (1)$$

where the coefficients c_i are determined by the abscissae x_i. Integration formulae of the Eq (1) is called Newton-cotes closed integration formulae. They are closed since the end points a and b are the extreme abscissae in the formulae.

On the Other hand, formulae which do not employ the end points are called Newton-cotes open integration formulae. These are as.

(a) $$\int_{x_0}^{x_2} y\,dx = 2hy_1 + \frac{h^3}{3} y''(\bar{x}), \quad (x_0 < \bar{x} < x_2)$$

(b) $$\int_{x_0}^{x_3} y\,dx = \frac{3h}{2}(y_1 + y_2) + \frac{3h^2}{4} y''(\bar{x}), \quad (x_0 < \bar{x} < x_3)$$

(c) $$\int_{x_0}^{x_4} y\,dx = \frac{4h}{3}(2y_1 - y_2 - 2y_3) + \frac{14}{45} h^5 y^{iv}(\bar{x}), \quad (x_0 < \bar{x} < x_4)$$

(d) $$\int_{x_0}^{x_5} y\,dx = \frac{5h}{24}(11y_1 + y_2 + y_3 + 11y_4) + \frac{95}{144} h^2 y^{iv}(\bar{x}), \quad (x_0 < \bar{x} < x_5)$$

(e) $\Rightarrow$ $$\int_{x_0}^{x_6} y\,dx = \frac{6h}{20}(11y_1 - 14y_2 + 26y_3 - 14y_4 + 11y_5) + \frac{41 h^7}{140} y^{vi}(\bar{x}), \quad (x_0 < \bar{x} < x_6)$$

Gaussian Integration :-

Let the numerical evalution of the integral

$$I = \int_a^b f(x)\,dx$$

In previous lecturers, we also derived some integration formulae which require values of the function at equal intervals. Gauss derived a formula which uses same function values but different spacing points of interval.

Gauss's formula is expressed in the form

$$\int_{-1}^{1} F(u)\,du = W_1 F(u_1) + W_2 F(u_2) + \dots W_n F(u_n)$$

$$= \sum_{i=1}^{n} W_i F(u_i) \quad \text{——— (1)}$$

where W_i and u_i are called the weights and abscissae, respectively. An advantage of this formula is that the abscissae and weights are symmetrical with respect to the middle point of interval.

$$\int_{-1}^{1} F(u)\,du = \int_{-1}^{1} (C_0 + C_1 u + C_2 u^2 + \dots C_{2n-1} u^{2n-1})\,du = 2C_0 + \frac{2}{3} C_1 + \frac{2}{5} C_2 \dots$$

* As an example, when $n = 1$, we solve $P_2(u) = 0$

i.e $\frac{1}{2}(3u^2 - 1) = 0 \Rightarrow u^2 = \frac{1}{3} \Rightarrow u = \pm\frac{1}{\sqrt{3}}$

which gives the two abscissae

$$u_0 = -\frac{1}{\sqrt{3}} \text{ and } u_1 = \frac{1}{\sqrt{3}}$$

The corresponding weights are given by

$$W_0 = \int_{-1}^{1} \frac{u - u_1}{u_0 - u_1}\,du = \frac{1}{u_0 - u_1}\left[\frac{u^2}{2} - u_1 \cdot u\right]_{-1}^{1}$$

$$W_0 = 1$$

and $W_1 = \int_{-1}^{1} \frac{u-u_0}{u_1-u_0} du = \frac{1}{u_0-u_1} \int_{-1}^{1} (u-u_0) du$

$$W_1 = \frac{1}{u_0-u_1} \left[\frac{u^3}{2} - u u_0 \right]_{-1}^{1} = 1$$

Similarly $n=3$, we solve $P_4(u)=0$. That is

$$\frac{1}{8}(35u^4 - 30u^2 + 3) = 0$$

which gives the four abscissae.

$$u_i = \pm \left(\frac{15 \pm 2\sqrt{30}}{35} \right)^{1/2}$$

The W_i weights can be obtaind from Equation

$$W_i = \int_{-1}^{1} \prod_{j=0, j\neq i}^{n} \frac{u-u_j}{u_i-u_j} du.$$

The u_i are the zeros of the $(n+1)^{th}$ Legendre polynomial $P_{n+1}(u)$ which can be generated using the recurrence relation.

$$(n+1) P_{n+1}(u) = (2n+1) u \cdot P_n(u) - n P_{n-1}(u)$$

where $P_0(u) = 1$ and $P_1(u) = u$ then the first five legendre polynomials are as.

$$P_0(u) = 1$$

$$P_1(u) = u$$

$$P_2(u) = \frac{1}{2}(3u^2 - 1)$$

$$P_3(u) = \frac{1}{2}(5u^3 - 3u)$$

$$P_4(u) = \frac{1}{8}(35u^4 - 30u^2 + 3)$$

In general case, the limits of the integral in Equation (i) $\int_a^b f(x)\,dx$ — ① have to changed to in equation (ii) $\int_{-1}^{1} F(u)\,du = W_1F(u_1) + W_2F(u_2) + \dots W_nF(u_n)$

$$= \sum_{i=1}^{n} W_i F(u_i) \quad — ②$$

by means of the transformation.

$$x = \frac{1}{2}u(b-a) + \frac{1}{2}(a+b) \quad — ③$$

Exp. Find $I = \int_0^1 x\,dx$ by Gauss's formula.

Sol. The first step is to change the limits by Eq ③

$$x = \frac{1}{2}u(b-a) + \frac{1}{2}(a+b)$$

$$= \frac{1}{2}u(1-0) + \frac{1}{2}(0+1)$$

$$x = \frac{1}{2}(u+1)$$

$x=0 \Rightarrow u=-1$

$x=1 \Rightarrow u=1$

$dx = \frac{1}{2}du$

This gives

$$I = \frac{1}{4}\int_{-1}^{1}(u+1)\,du$$

$$I = \frac{1}{4}\int_{-1}^{1}(u+1)\,du = \frac{1}{4}\sum_{i=1}^{n} W_i F(u_i)$$

where $F(u_i) = u_i + 1$

We take $n=4$ and using abscissae and weights corresponding to $n=4$ from Gaussian integration table.

$$I = \frac{1}{4}\Big[(-0.86114+1)(0.34785) + (-0.33998+1)\times(0.65214) + (0.33998+1)(0.65214) + (0.86114+1)(0.34785)\Big]$$

$$= 0.49999$$