# Abstract Algebra

## GALOIS THEORY :-

"Galois theory" is the one of the most eligent theories in Abstract Algebra. This theory is an excellent Composite of the theory of groups with the theory of algbraic field extensions and with has a very important application in the theory of equations.

The Galois theory, when reduced to its bare essentials, is a study of some of the relationships of three mathematical objects such as an irreducible polynomial $f(x) \in F[x]$ for some field $F$, and the splitting field $K$ of $f(x)$ and the set of automorphisms of $K$ over $F$.

## AUTOMORPHISMS OF FIELDS

**Definition 1** :- Let $F$ be an arbitrary field. A mapping $\sigma$ of $F$ onto itself such that $\sigma(a+b) = \sigma(a) + \sigma(b)$ and $\sigma(ab) = \sigma(a)\sigma(b)$ for all $a, b \in F$, is Known as automorphism of $F$. The set of all automorphism of F is denoted by $A(F)$.

**Definition 2** :- Two automorphisms $\sigma$ and $\tau$ of $F$ are said to be distinct if $\sigma(a) \neq \tau(a)$ for some $a \in F$.

**Notes** :-

(i) The set $A(F)$ of all automorphism of $F$ forms a group with respect to the composite composition.

## FIXED FIELD OF A GROUP OF AUTOMORPHISMS

**Definition :-** Let $G$ be a subgroup of the automorphism group of a field $F$. Let $F_0$ be the subset of $F$ consisting of those elements of $F$ which are left fixed by every automorphism in $G$. That is,

$$F_0 = \{a \in F : \sigma(a) = a \ \forall \ \sigma \in G\}$$

Then $F_0$ is called the fixed field of $G$.

**Lemma 1 :-** The fixed field of $G$ is a subfield of $F$.

**Proof :-** By above definition of fixed field of $G$, we have

$$F_0 = \{a \in F : \sigma(a) = a \ \forall \ \sigma \in G\}$$

Let $a, b \in F_0$, so $a, b \in F$ such that

$$\sigma(a) = a \text{ and } \sigma(b) = b$$

since $F$ is a field, then

For, $a, b \in F \implies a \pm b \in F$, $ab \in F$

Now $\sigma(a-b) = \sigma(a) - \sigma(b) = a - b$

$\therefore a - b \in F_0$ as $a - b \in F$

Let $0 \neq b \in F$, then $b^{-1} \in F$

Now $\sigma(ab^{-1}) = \sigma(a)\,\sigma(b^{-1}) = \sigma(a)(\sigma(b))^{-1}$

$[\because \sigma(b^{-1}) = (\sigma(b))^{-1}]$

$\sigma(ab^{-1}) = ab^{-1}$

$\therefore ab^{-1} \in F_0$ because $ab^{-1} \in F$

Hence for $a, b \in F_0$, $a - b \in F_0$ and $ab^{-1} \in F_0$

Consequently $F_0$ is a subfield of $F$.