GROUP THEORY

Theorem :- Fundamental theorem on homomorphism of groups :- "Every homomorphic image of a group G is isomorphic to some quotient group of G."

Proof :- If G' be a homomorphic image of G and f be the homomorphism of G onto G'.

Let $K = \ker f$

If $a \in G$, then $Ka \in G/K$ and $f(a) \in G'$.

Define a mapping

$$\phi : G/K \to G' \text{ such that}$$

$$\phi(Ka) = f(a) \quad \forall\ a \in G.$$

ϕ is well defined. Now we show that ϕ is well defined.

i.e. if $a, b \in G$ and $Ka = Kb$, then

$$\phi(Ka) = \phi(Kb).$$

Then $Ka = Kb \Rightarrow ab^{-1} \in K$

$\Rightarrow f(ab^{-1}) = e' \Rightarrow f(a) f(b^{-1}) = e'$

$\Rightarrow f(a)[f(b)]^{-1} = e' \Rightarrow f(a)[f(b)]^{-1} \cdot f(b) = e' f(b)$

$\Rightarrow f(a) e' = f(b) \Rightarrow f(a) = f(b)$

$\Rightarrow \phi(Ka) = \phi(Kb)$

$\Rightarrow \phi$ is well defined.

ϕ is one-one. Now, we have

$$\phi(Ka) = \phi(Kb) \Rightarrow f(a) = f(b)$$

$\Rightarrow f(a)[f(b)]^{-1} = f(b)[f(b)]^{-1}$

$\Rightarrow f(a) \cdot f(b^{-1}) = e' \Rightarrow f(ab^{-1}) = e'$

$\Rightarrow ab^{-1} \in K \quad (\because K \text{ is Kernel})$

$\Rightarrow Ka = Kb$

$\therefore \phi$ is one-one.

ϕ is onto. Let $y \in G'$, then $y = f(a)$ for some $a \in G$ as f is onto G'.

Now $Ka \in G/K$ and we have

$\phi(Ka) = f(a) = y$

$\Rightarrow \phi$ is onto.

Therefore, ϕ is an isomorphism of G/K onto G'.

Hence $G/K \cong G'$.

Notes :-

(i) According to famous Mathematician JACOBIAN, the theorem can be stated as follows:

"Any factor group of G is a homomorphic image of G and Conversely every homomorphic image of group G is isomorphic to a quotient group of G".

(ii) The fundamental theorem on homomorphism of group tells us how to find all possible homomorphic image of given group G.

(iii) Except for isomorphism, these homomorphic images must be expressible, in the form G/N where N is normal in G.

(iv) For any normal subgroup N of G, G/N is a homomorphic image of G. Thus we have a one-one correspondence between the normal subgroups of G and the homomorphic image of G.