Fundamental theorem of Arithmetic

Corollary :- Any positive integer $n > 1$ can be written uniquely in a canonical form

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$

where for $i = 1, 2, \ldots, r$ each k_i is a positive integer and each p_i is a prime, with $p_1 < p_2 < \cdots < p_r$.

To illustrate, the canonical form of the integer 360 is $360 = 2^3 \cdot 3^2 \cdot 5$. As further examples

$$4725 = 3^3 \cdot 5^2 \cdot 7 \quad \text{and} \quad 17640 = 2^3 \cdot 3^2 \cdot 5 \cdot 7^2$$

Prime factorizations provide another means of calculating greatest common divisors. For suppose that $p_1, p_2, \ldots, p_n$ are the distinct primes that divide either of a or b. Allowing zero exponents, we can write

$$a = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}, \quad b = p_1^{j_1} p_2^{j_2} \cdots p_n^{j_n}$$

Then

$$\gcd(a, b) = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$$

where $r_i = \min(k_i, j_i)$, the smaller of the two exponents associated with p_i in the two representations. In the case $a = 4725$ and $b = 17640$, we would have

$$4725 = 2^0 \cdot 3^3 \cdot 5^2 \cdot 7, \quad 17640 = 2^3 \cdot 3^2 \cdot 5 \cdot 7^2$$

and so

$$\gcd(4925, 17640) = 2^0 \cdot 3^2 \cdot 5 \cdot 7$$
$$= 315.$$

(b) $\gcd(24, 138) = 24x + 138y$

$138 = 24 \times 5 + 18$ — (i)

$24 = 18 \times 1 + 6$ — (ii)

$18 = 6 \times 3 + 0$ — (iii)

$$24 \overline{)\,138\,}(5$$
$$120$$
$$18 \overline{)\,24\,}(1$$
$$18$$
$$6 \overline{)\,18\,}(3$$
$$18$$
$$\times\times$$

HCF of $(24, 138) = 6$

From (ii)

$24 = 18 \times 1 + 6$

$6 = 24 - 18 \times 1$

$6 = 24 - (138 - 120) \times 1$

$6 = 24 - 138 \times 1 + 120 \times 1$

$6 = 24 + 120 \times 1 + 138 \times (-1)$

$6 = 24 + 24 \times 5 \times 1 + 138 \times (-1)$

$6 = 24(1+5) + 138 \times (-1)$

$6 = 24 \times 6 + 138 \times (-1)$

$\therefore 6 = 24x + 138y$

$x = 6, \quad y = -1$

$17 \times 7 = 119$

$34 \times 3 = 102$

(c) $\gcd(119, 272) = 119x + 272y$

$272 = 119 \times 2 + 34$ — (i)

$119 = 34 \times 3 + 17$. — (ii)

$34 = 17 \times 2 + 0$ — (iii)

$$119 \overline{)\,272\,}(2$$
$$238$$
$$34 \overline{)\,119\,}(3$$
$$102$$
$$17 \overline{)\,34\,}(2$$
$$34$$
$$\times\times\times$$

HCF of $(1119, 72) = 17$.

From (ii)

$119 = 34 \times 3 + 17$

$17 = 119 - 34 \times 3$

(2) (1) Use the Euclidean Algorithm to obtain integers x and y satisfying the following

(a) $\gcd(56, 72) = 56x + 72y$

$72 = 56 \times 1 + 16$ — (i)

$56 = 16 \times 3 + 8$ — (ii)

$16 = 8 \times 2 + 0$ — (iii)

$$
\begin{array}{r}
56\overline{)72}(1 \\
56 \\
\hline
16\overline{)56}(3 \\
48 \\
\hline
8\overline{)16}(2 \\
16 \\
\hline
\times\times
\end{array}
$$

HCF of $(56, 72) = 8$

From (ii)

$56 = 16 \times 3 + 8$

$8 = 56 - 16 \times 3$

$8 = 56 - (72 - 56) \times 3$

$8 = 56 - 72 \times 3 + 56 \times 3$

$8 = 56 + 56 \times 3 + 72 \times (-3)$

$8 = 56 (1 + 3) + 72 \times (-3)$

$8 = 56 \times 4 + 72 \times (-3)$

$\therefore 8 = 56x + 72y$

when $x = 4$, and $y = -3$