Abstract Algebra

Definition 3 A left R-module M is said to be faithful iff the representation of R associated with M is injective.

Theorem:- Let R be a ring and let I a minimal left ideal of R. If M is a faithful simple left R-module, then M is isomorphic to the left R module I.

Proof:- Let $0 \neq a_0 \in I$. Since M is faithful, there is some $x \in R$ such that $a_0 x \neq 0$ and the map $\rho_x : I \to R$ defined by $\rho_x(a) = ax \ (a \in I)$ is a non-zero homomorphism.

By Schur's lemma, ρ_x is a monomorphism since the R-module I is simple, and ρ_x is an epimorphism because the R-module M is simple.

Definition 4:- An R-module M is said to be semi-simple if it can be expressed as a direct sum of simple submodules.

Notes:- ① The following conditions on an R-module M are equivalent

(i) M is a sum of simple submodules

(ii) M is a semisimple module

(iii) Every submodule of M is a direct summand of M.

② Every submodule and every homomorphic image of a semi-simple module is semi-simple.

③ If M is a semi-simple R-module and it is equal to the direct sum of family $\{M_i : i \in \Lambda\}$ of simple submodule, then every simple submodule of M is Hence to one of the M_i.

FREE MODULES

Definition 1 (Basis of Module). Let M be a module over a ring R with unity and let S be a subset of M. Then S is called a basis of M is

1. S generates M
2. and S is linearly independent.

If S is a basis of M, then in particular, $M \neq \{0\}$ and if $R \neq \{0\}$, then every element of M can be uniquely expressed as a linear combination of elements of S.

Definition 2 An R-module M is called a free module if there exists a subset S of M such that S generates M and S is linearly independent set.

For example:- If $M = (0)$, then it is free module because the empty set is its basis.

Definition 3 Let M be a finitely generated free module over a commutative ring R with unit element. The number of elements in any basis of M is known as the rank of M which is written as rank(M) or rank$_R(M)$.

If R is a field, then M is regarded as a vector space, then rank(M) is known as the dimension of M, which is denoted by dim(M).

Definition 4. A ring R has an invariant rank property if for every free R-module M, the rank of M over R is defined i.e. any two basis of M has the same cardinality.

Theorem:- Let R be a ring with the invarient rank property and let M and N be free R-modules. Then $M \cong N$ iff rank (M) = rank (N).

Proof:- Let us first suppose that $M \cong N$. Let $\phi : M \rightarrow N$ be an R-module isomorphism. Let B be a basis of M, then clearly $\phi(B)$ is a basis of N because ϕ is bijective mapping.

Also cardinality of B is equal to the cardinality of $\phi(B)$. Hence

$$\text{rank}(M) = \text{rank}(N)$$

Conversely, let us suppose that rank(M) = rank(N). Let B and B' be the basis of M and N respectively. Then cardinality of B = Cardinality of B'. Thus there is a one-one onto mapping $f : B \rightarrow B'$. Now there exists an R-module homomorphism $\phi : M \rightarrow N$ such that $\phi_{|B} = f$. Obviously ϕ is one-one and onto. Hence ϕ is an isomorphism. Consequently $M \cong N$.