Abstract Algebra

FIELD

A non-empty set F is said to be a field if F is a Commutation ring with unit element and which each non-zero element has its multiplicative inverse.

Example :- The set of real numbers R is a field.

SUBFIELD

A non-empty subset S of a field F is said to a subfield of F if it is closed under addition and multiplication and has a field structure relatives to the induced Composition.

Notes :-

(i) If a subfield S of a field F is not equal to F, then S is called a proper subfield of F.

S has at least two elements namely 0 and 1.

Theorem :- The necessary and sufficient Conditions for a non-empty subset S of a field F to be a subfield of a field F are that

(i) $a \in S,\ b \in S \Rightarrow a-b \in S$

(ii) $a \in S,\ b \in S \Rightarrow ab^{-1} \in S$

Proof :- Necessary Condition

Suppose S is a subfield of a field F, then S is closed under addition and multiplication. Since S is a subgroup under addition so Condition (i) is obtained and S is a field itself, so every non-zero element of S has its multiplicative inverse, therefore Condition

(ii) is then obtained.

Sufficient Condition

(i) and (ii) are given. Suppose the Condition S is subfield we have to prove

From (i) $a \in S$, $a \in S \Rightarrow a - a \in S \Rightarrow 0 \in S$

So identity of addition exists.

Further $a \in S$, $a \in S \Rightarrow 0 - a \in S \Rightarrow -a \in S$

Thus additive inverse of elements exist.

Again from (i)

$$a \in S, -b \in S \Rightarrow a - (-b) \in S$$
$$\Rightarrow a + b \in S$$

So S is closed under addition

Since associative and Commutativity are always obvious.

Hence S is an abelian group under addition

From (ii) $a \in S$, $a \in S \Rightarrow aa^{-1} \in S \Rightarrow 1 \in S$

Therefore, unit element exists in S.

Further since, if $a \neq 0$

$$a \in S, 1 \in S \Rightarrow 1 \cdot a^{-1} \in S \Rightarrow a^{-1} \in S$$

Thus inverse of non-zero elements exists in S.

From (ii)

$$a \in S, b^{-1} \in S \Rightarrow ab \in S$$

or, $a \in S$, $b \in S \Rightarrow ab \in S$

That is, S is closed under multiplication

And associativity and distributive laws hold in S. Hence S is a field. that is S is a subfield of F.