LINEAR PROGRAMMING

Theorem 4:- The set of all feasible solutions of a L.P.P. is a Convex set

Proof :- Let X be the set of all feasible solutions of a L.P.P. -

$$Ax = b, x \geq 0 \quad \text{——} \quad (1)$$

Case I If the set X has only one element. then X is Convex set. Hence the theorem is true in this case.

Case II If the set X has at least two elements

Let x_1 and x_2 be any two distinct elements in X.

$$\therefore \quad Ax_1 = b, \qquad x_1 \geq 0$$
$$\text{and} \quad Ax_2 = b, \qquad x_2 \geq 0$$

If $x_3 = \lambda x_1 + (1-\lambda) x_2, \quad 0 \leq \lambda \leq 1$

then $Ax_3 = A\lambda x_1 + (1-\lambda) Ax_2$

$= \lambda b + (1-\lambda) b = b$

Also since $x_1 \geq 0$, $x_2 \geq 0$, $\lambda \geq 0$, $1-\lambda \geq 0$, as $\leq \lambda \leq 1$.

$\therefore x_2 = \lambda x_1 + (1-\lambda) x_2 \geq 0$

i.e. x_2 satisfy (1). Thus $x_2 = \lambda x_1 + (1-\lambda) x_2$ is also a F.S. and so belongs to set X.

But x_2 is a Convex Combination of any two distinct points x_1 and x_2 in X.

Hence by definition the set X is a Convex set.

Note :- Since the convex combination of two points are infinite in number so from the above theorem we conclude that if a given L.P.P. has two feasible solution, then it has infinite number of feasible solutions.

Theorem 5 :- Every basic feasible solutions of the system $Ax = b$ $x \geq 0$ is an extreme point of the convex set of feasible solutions and conversely.

Proof :- To prove that every B.F.S. is an extreme point of the convex set of feasible solutions.

Let x be a B.F.S. of $Ax = b$ which is a n-component vector containing both zero (non-basic) and non-zero (basic) variables.

Let x_B and B be the vector of m basic variables and the matrix of vectors associated to basic variables in the B.F.S. x respectively then

$$x = \{x_B, 0\} \quad \text{——— (1)}$$

where 0 is a null vector of $(n-m)$ components.

and $$Ax = b \Rightarrow B \cdot x_B = b \quad \text{——— (2)}$$

Now we have to prove that x is an extreme points.

We shall prove this by using contradiction.

If x is not an extreme point then there exist two distinct points

x_1 and x_2 in X such that

$$x = \lambda x_1 + (1-\lambda)x_2, \quad 0 < \lambda < 1 \quad \text{---} (3)$$

But x_1 and x_2 can be expressed as

$$x_1 = [u_2, v_1] \text{ and } x_2 = [u_2, v_2] \quad \text{---} (4)$$

where u_1 and u_2 are vectors of m basic variables of x_1 and x_2 respectively and v_1 and v_2 are $(n-m)$ component vectors.

Substituting the value of x and x_1, x_2 from (1) and (4) in (3), we have

$$[x_B, 0] = \lambda[u_1, v_1] + (1-\lambda)[u_2, v_2], \quad 0 < \lambda < 1$$

$$= \{\lambda u_1 + (1-\lambda)v_2, \ \lambda v_1 + (1-\lambda)v_2] \quad \text{---} (5)$$

$$0 < \lambda < 1$$

since $\lambda > 0$, $1-\lambda > 0$, $v_1 \geq 0$ and $v_2 \geq 0$

The relation (5) can only be satisfied when $v_1 = 0$ and $v_2 = 0$

$$\therefore x_1 = [u_1, 0], \quad x_2 = [u_2, 0]$$

since x_1 and x_2 are in X, therefore from (2) we have

$$\because Ax_1 = Bu_1 = b \text{ and } Ax_2 = Bu_2 = b$$

$$\text{i.e. } b = Bx_B = Bu_1 = Bu_2$$

which gives $x_B = u_1 = u_2$

$$\therefore x = x_1 = x_2$$

which is contradiction to the assumption $x_1 \neq x$ i.e. x cannot be expressed as a convex combination of any two distinct points in the set of all feasible solutions.

Hence x must be an extreme point.