# Abstract Algebra

**Corollary** (Factor theorem) Let $K$ be an extension of a field $F$ and $\alpha \in K$ be a root of a polynomial $f(x) \in F(x)$. then in $K[x]$, $f(x) = (x-\alpha)\, q(x)$, where $q(\alpha) \neq 0$

**Proof** :- Since $\alpha \in K$ is a root of a polynomial $f(x) \in F[x]$, then $f(\alpha) = 0$.

By Remainder theorem, Corresponding to an element $\alpha \in K$ and $\alpha \in K$ and $f(x) \in F[x]$, there exists a polynomial $q(x) \in K[x]$, such that

$$f(x) = (x-\alpha)q(x) + f(\alpha) \Rightarrow f(x) = (x-\alpha)q(x)$$

$$[f(\alpha) = 0]$$

Hence the Corollary.

**Notes** :-

(i) A polynomial of degree $n$ over a field can have at most $n$ roots in any extension field.

(ii) **Kroncker's theorem** . Let $f(x)$ be an irreducible polynomial of positive degree in $F[x]$. Then there exists an extension $K$ of $F$ in which $f(x)$ has a root and also

$$[K : F] = \deg . f(x).$$

**Proof** :- Let $\deg . f(x) = n$. since $f(x)$ is irreducible in $F[x]$, then $(f(x))$ is a maximal ideal of $F[x]$, therefore $F[x]/(f(x))$ is a field which is an extension of $F$.

Let $K = F(x)/f(x)$ and let $K' = \{(f(x)) + a : a \in F\}$ a subset of $K$.

First we show that $K'$ is a subfield of $K$.

Let $(f(x))+a$ and $f(x)+b$ be any two elements of $k'$ such that $(f(x))+b \neq (f(x))$.
Since $k' \subset K$ and $(f(x))+b \in K$, then
$((f(x))+b)^{-1} \in K$

$$\Rightarrow (f(x))+b^{-1} \in K \Rightarrow b^{-1} \in F \text{ as } b \neq 0$$

So $((f(x))+a)((f(x))+b)^{-1} = ((f(x))+a)((f(x))+b^{-1})$

$$= (f(x))+ab^{-1} \in k'$$

$$[\because ab^{-1} \in F]$$

Thus $k'$ is a subfield of $K$.

Now define a mapping $\phi : F \to k'$
By $\phi(a) = (f(x))+a \ \forall a \in F$.

Then we show that $\phi$ is one-one onto homomorphism.

$\phi$ is one-one. Let $a, b \in F$ and assume that

$$\phi(a) = \phi(b) \Rightarrow (f(x))+a = (f(x))+b$$

$$\Rightarrow a-b \in (f(x))$$

$$\Rightarrow a-b = f(x)\,g(x) \text{ for some } g(x) \in F[x]$$

Since $f(x)$ being irreducible polynomial. So its product with any other non-zero polynomial can not be equal to a constant polynomial.

$$\therefore g(x) = 0 \Rightarrow a-b=0 \Rightarrow a=b$$

Thus $\phi$ is one-one.

$\phi$ is onto. Let $f(x)+a \in K'$ such that $a \in F$, then

$$\phi(a) = (f(x))+a \ \forall a \in F$$

clearly $\phi$ is onto.