# Dimension of a finitely generated (or finite dimensional) Vector space:-

The number of elements in any basis of a finite dimensional vector space $V(F)$ is called the dimension of the vector space $V(F)$ and is denoted by $\dim(V)$.

Exp. The dimension of vector space $V_n(F)$ or $V(F^n)$ is $n$.

i.e. $\dim V_n(F) = n$ or $\dim V(F^n) = n$

* If a field $F$ is regarded as a vector space over $F$ i.e. $V(F)$, then $F$ will be of dimension $1$ ($\dim V(F) = 1$) and the set $S = \{(1)\}$.

## Some Properties of finite dimensional vector Space

**Theorem-1** Every linearly independent subset of a finitely generated vector space $V(F)$ forms a part of a basis of $V$.

or- Every linearly independent subset of a finitely generated vector space $V(F)$ is either a basis of $V$ or can be extended to form a basis of $V$.

**Theorem 2-** Each set of $(n+1)$ or more vectors of a finite dimensional vector space $V(F)$ of dimension $n$ is linearly dependent.

**Theorem 3-** Every finite dimensional vector space $V(F)$ has a basis.

or. If $V(F)$ is a finite dimensional vector space of dimension $n$, then any set of $n$ linearly independent vectors in $V$ forms a basis of $V$.

Finite Dimensional Vector Space (FDVS) -

→ vector space V has finite basis of vectors.

→ Dimension of V has finite basis consisting having n vectors as denoted

$\dim V(F) = n$ F is R or C

→ Basis of vector space (V) is a set of vectors which are linearly Indep. and linearly spanning of V.

* i.e A vector space V(F) has a finite basis consisting of n number of vectors then V is FDVS, otherwise V is called infinite dimensional vector space.

<u>Theorem</u> - If $S = \{v_1, v_2, \ldots v_n\}$ is basis of V, then every set containg more than n vectors in V is linearly dependent.

Proof - Let $T = \{u_1, u_2 \cdots u_m\}$ be any set of m vectors in V where $m > n$

To show that T is L.D.

Exp. $S = \{\overset{v_1}{(1,3,-1)}, \overset{v_2}{(1,1,0)}, \overset{v_3}{(3,3,0)}, \overset{v_4}{(5,9,-1)}\}$ must be L.D

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$ no. of vector $m = 4$

∴ no of scalars > no. of Eq. then it has nontrivial solu. i.e $\alpha_i \neq 0$

no. of dim $n = 3$

$m > n$

then we need to find scalars $\alpha_1, \alpha_2 \ldots \alpha_m$ (not all zero) such that linear combination of T.

$\alpha_1 u_1 + \alpha_2 u_2 + \cdots \alpha_m u_m = 0$ —— (1)

Because $S = \{v_1, v_2 \ldots v_n\}$ is basis for V, it follows that each $u_i$ is a linear combination of S. Then

$u_1 = \beta_{11} v_1 + \beta_{21} v_2 + \cdots \beta_{n1} v_n$

$u_2 = \beta_{12} v_1 + \beta_{22} v_2 + \cdots \beta_{n2} v_n$

$\vdots$

$u_m = \beta_{1m} v_1 + \beta_{2m} v_2 + \cdots \beta_{nm} v_n$

i.e $u_i = \beta_{1i} v_1 + \beta_{2i} v_2 + \cdots \beta_{ni} v_n$

Substituting each $u_i$ in Eqn (1), we get

$$\alpha_1(\beta_{11}v_1 + \beta_{21}v_2 + \cdots \beta_{n1}v_n) + \alpha_2(\beta_{12}v_1 + \beta_{22}v_2 + \cdots \beta_{n2}v_n) +$$

$$\cdots + \alpha_m(\beta_{1m}v_1 + \beta_{2m}v_2 + \cdots \beta_{nm}v_n) = 0$$

The set of vectors $S = \{v_1, v_2 \cdots v_n\}$ is basis for V.
So. $v_1, v_2 \cdots v_n$ are L.I.

$$(\alpha_1\beta_{11} + \alpha_2\beta_{12} + \alpha_3\beta_{13} + \cdots \alpha_m\beta_{1m})v_1 + (\alpha_1\beta_{21} + \alpha_2\beta_{22} + \alpha_3\beta_{23} + \cdots \alpha_m\beta_{2m})v_2$$

$$+ \cdots (\alpha_1\beta_{n1} + \alpha_2\beta_{n2} + \cdots \alpha_m\beta_{nm})v_n = 0$$

$v_1, v_2, v_3 \cdots v_n$ are L. Indp. vectors.

$$\Rightarrow \left.\begin{aligned} &\alpha_1\beta_{11} + \alpha_2\beta_{12} + \alpha_3\beta_{13} + \cdots \alpha_m\beta_{1m} = 0 \\ &\alpha_1\beta_{21} + \alpha_2\beta_{22} + \alpha_3\beta_{23} + \cdots \alpha_m\beta_{2m} = 0 \\ &\cdots \\ &\cdots \\ &\alpha_1\beta_{n1} + \alpha_2\beta_{n2} + \alpha_3\beta_{n3} + \cdots \alpha_m\beta_{nm} = 0 \end{aligned}\right\}$$

This system is homogeneous linear equations in m variables i.e variables (m) > no. of Eqn. (n). Then this system has non-zero solutions or non-trivial sol.
So. S is L.D.
i.e $\alpha_i \neq 0$ all scalars $\alpha_1, \alpha_2 \cdots \alpha_m$ are not zero.