# GROUP THEORY



## EXTERNAL DIRECT PRODUCTS OF GROUPS

Let $G_1, G_2$ be two arbitrary given groups, then we can define a composition in the product set $G_1 \times G_2$ relative to which it acquires a group structure.

Let $(a_1, a_2) \in G_1 \times G_2$ and $(b_1, b_2) \in G_1 \times G_2$ so that $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$.

Now the set $G_1 \times G_2 = \{(x, y) : x \in G_1, y \in G_2\}$ together with the binary operation given by

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1, a_2 b_2)$$

forms a group. This group is known as the External direct product group of $G_1$ and $G_2$.

To show that $G_1 \times G_2$ forms a group under multiplication defined above.

**Proof :-** (i) closure property :- Let $(a_1, a_2)$ $(b_1, b_2) \in G_1 \times G_2$ so that $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$. Since $G_1$ and $G_2$ both are groups under multiplication, then by closure property in $G_1$ and $G_2$, we have

$$a_1 b_1 \in G_1 \;\forall\; a_1, b_1 \in G_1$$

$$\text{and } a_2 b_2 \in G_2 \;\forall\; a_2, b_2 \in G_2$$

$$\therefore (a_1 b_1, a_2 b_2) \in G_1 \times G_2$$

$$\Rightarrow (a_1, a_2) \cdot (b_1, b_2) \in G_1 \times G_2$$

Thus $G_1 \times G_2$ is closed under multiplication.

(ii) Associative property :- For any elements $(a_1, a_2)(b_1, b_2)$ and $(c_1, c_2)$ of $G_1 \times G_2$. we have $a_1, b_1, c_1 \in G_1$ and $a_2, b_2, c_2 \in G_2$.

Since $G_1, G_2$ are groups so by associative law in $G_1$ and $G_2$, we have

$(a_1 b_1) c_1 = a_1(b_1 c_1)$ and $(a_2 b_2) c_2 = a_2(b_2 c_2)$

Now $[(a_1, a_2)(b_1, b_2)](c_1, c_2) = \{(a_1 b_1, a_2 b_2)\}(c_1, c_2)$

$$= \{(a_1 b_1) c_1, (a_2 b_2) c_2\}$$
$$= \{a_1 (b_1 c_1), a_2 (b_2 c_2)\}$$
$$= (a_1, a_2) \{(b_1 c_1, b_2 c_2)\}$$
$$= (a_1, a_2) \{(b_1, b_2)(c_1, c_2)\}$$

This shows that $G_1 \times G_2$ is associative.

(iii) Existence of identity :- Let $e_1, e_2$ be the identities elements in $G_1$ and $G_2$ respectively so that $(e_1, e_2) \in G_1 \times G_2$

Moreover, for any $(a_1, a_2) \in G_1 \times G_2$, we have

$(a_1, a_2)(e_1, e_2) = (a_1 e_1, a_2 e_2) = (a_1, a_2)$

$(\because a_1 e_1 = G_1, a_2 e_2 = G_2)$

and $(e_1, e_2)(a_1, a_2) = (e_1 a_1, e_2 a_2) = (a_1, a_2)$

$(\because e_1 a_1 = a_1, e_2 a_2 = a_2)$

$\therefore (a_1, a_2)(e_1, e_2)$ is the identity element of $G_1 \times G_2$.

(iv) Existence of inverse :- Since $G_1$ and $G_2$ are groups, so for every $a_1 \in G_1$, $a_1^{-1} \in G_1$ and for every $a_2 \in G_2$, $a_2^{-1} \in G_2$.

$\therefore (a_1, a_2) \in G_1 \times G_2$ and $(a_1^{-1}, a_2^{-1}) \in G_1 \times G_2$

Moreover,

$$(a_1, a_2)(a_1^{-1}, a_2^{-1}) = (a_1 a_1^{-1}, a_2 a_2^{-1}) = (e_1, e_2)$$

$$[\because a_1 a_1^{-1} = e_1,\ a_2 a_2^{-1} = e_2]$$

Also, $(a_1^{-1}, a_2^{-1})(a_1, a_2) = (a_1^{-1} a_1, a_2^{-1} a_2) = (e_1, e_2)$

$$\therefore (a_1, a_2)(a_1^{-1}, a_2^{-1}) = (e_1, e_2) = (a_1^{-1}, a_2^{-1})(a_1, a_2)$$

$$\forall (a_1, a_2) \in G_1 \times G_2.$$

This shows every element $(a_1, a_2) \in G_1 \times G_2$ has its inverse in $G_1 \times G_2$.

Hence $G_1 \times G_2$ forms a group under the above defined operation.