Expansion of $(1+x)^n$ when $n$ is either negative or fractional.

Let $f(x) = (1+x)^n \Rightarrow f(0) = 1$

$f'(x) = n(1+x)^{n-1} \Rightarrow f'(0) = n$

$f''(x) = n(n-1)(1+x)^{n-2} \Rightarrow f''(0) = n(n-1)$

$\cdots$

$\cdots$

$f^n(x) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \Rightarrow f^n(0) = \lfloor n$

from Maclaurin's Theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots$$

$$= 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \frac{n(n-1)(n-2)}{\lfloor 3} x^3 + \cdots$$

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \cdots$

(ii) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1\cdot 2} x^2 + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3} x^3 + \cdots$

Evaluate

$$\lim_{x\to 0} (1+x)^{1/x} = \lim_{x\to 0} \left[ 1 + \frac{1}{x}\cdot x + \frac{\frac{1}{x}\cdot\left(\frac{1}{x}-1\right)}{\lfloor 2}\cdot x^2 + \frac{\frac{1}{x}\cdot\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{\lfloor 3}\cdot x^3 + \cdots \right]$$

$$= \lim_{x\to 0} \left[ 1 + 1 + \frac{1\cdot(1-x)}{\lfloor 2} + \frac{1\cdot(1-x)(1-2x)}{\lfloor 3} \cdots \right]$$

$$= \lim_{x\to 0} \left[ 1 + 1 + \frac{1(1-x)}{\lfloor 2} + \frac{1(1-x)(1-2x)}{\lfloor 3} \cdots \right]$$

$$\lim_{x\to 0} (1+x)^{1/x} = \left(1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots\cdots\right) = e$$

$\therefore e^x = \left(1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\cdots\right)$

$e^1 = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots$

Imp. **Inderterminate form**

Exp. Evaluate $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$

Solution - Given that $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$

when applying limit $x = 0$ then expression is - $\frac{e-e}{0} = \frac{0}{0}$ indeterminate form

Now, applying L. Hospital rule

$$\lim_{x \to 0} \frac{\frac{d}{dx}[(1+x)^{1/x} - e]}{\frac{d}{dx}(x)} = \lim_{x \to 0} \frac{\frac{d}{dx}[(1+x)^{1/x}]}{1} \quad \text{--- (1)}$$

Let $u = (1+x)^{1/x}$

$$\log u = \frac{1}{x} \log(1+x)$$

$$\frac{d}{dx}(\log u) = \frac{d}{dx}\left(\frac{1}{x} \log(1+x)\right)$$

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{x} \cdot \frac{1}{(1+x)} + \log(1+x) \cdot (-x^{-2})$$

$$= \frac{1}{x}[(1+x)^{-1}] - \frac{1}{x^2} \log(1+x)$$

$$= \frac{1}{x}(1 - x + x^2 - x^3 + x^4 \cdots) - \frac{1}{x^2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots\right)$$

$$= \left(\frac{1}{x} - 1 + x - x^2 + x^3 \cdots\right) - \left(\frac{1}{x} - \frac{1}{2} + \frac{x}{3} - \frac{x^2}{4} + \cdots\right)$$

$$\frac{1}{u}\frac{du}{dx} = \left(-1 + \frac{1}{2}\right) + x\left(1 - \frac{1}{3}\right) - x^2\left(1 - \frac{1}{4}\right) + \cdots$$

$$= \frac{1}{2} + \frac{2}{3}x - \frac{3}{4}x^2 + \frac{4}{5}x^3 - \cdots$$

$$\frac{du}{dx} = u\left(\frac{1}{2} + \frac{2}{3}x - \frac{3}{4}x^2 + \frac{4}{5}x^3 - \cdots\right)$$

$$\frac{du}{dx} = (1+x)^{1/x}\left(\frac{1}{2} + \frac{2}{3}x - \frac{3}{4}x^2 + \cdots\right) \quad \text{--- (2)}$$

Therefore required limit from (1)

$$\lim_{x \to 0} (1+x)^{1/x}\left(\frac{1}{2} + \frac{2}{3}x - \frac{3}{4}x^2 + \cdots\right)$$

$$\lim_{x \to 0} (1+x)^{1/x} \cdot \lim_{x \to} \left(\frac{1}{2} + \frac{2}{3}x - \frac{3}{4}x^2 + \cdots\right)$$

$$= e \times \frac{1}{2} = e/2 \quad \text{Hence.}$$